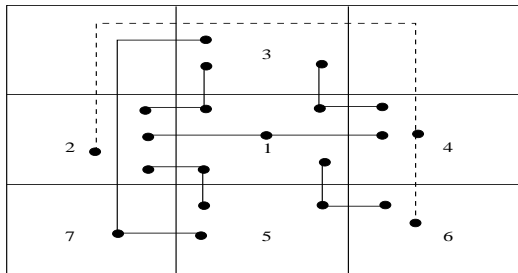


Extra Set Paradoxes

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We consider hypergraph games where two players alternately choose elements of the vertex set, the winner being the first to occupy an entire hyperedge. For example, the popular game of Tic-Tac-Toe corresponds to the vertex set $\{1, 2, \dots, 9\}$ and edge set $\{(1, 2, 3), (4, 5, 6), (7, 8, 9), (1, 4, 7), (2, 5, 8), (3, 6, 9), (1, 5, 9), (3, 5, 7)\}$. Let P_1 and P_2 denote the first and second player respectively. It can be shown by a strategy stealing argument that P_2 cannot force a win. We give two examples of the so-called extra set paradox for 3-uniform hypergraphs, where P_1 can win on the entire hypergraph, but not on a proper (edge or vertex induced) subgraph. Our example for the edge-induced case is minimal, and the existence of the vertex-induced case for uniform hypergraphs was an open question, albeit widely believed.

Consider the hypergraph $H = (V, E)$ where $V = \{1, 2, \dots, 7\}$ and $E = \{(1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 3, 4), (1, 5, 6), (3, 5, 7)\}$.



P_1 can force a win on this hypergraph as follows:

Move 1: P_1 picks 1. P_2 is forced to pick 2, for otherwise P_1 will pick 2 and seal the game, since at most two edges can be blocked in two moves.

Move 2: P_1 picks 3. P_2 is forced to pick 4 (immediate threat).

Move 3: P_1 picks 5. P_2 is forced to pick 6 (immediate threat).

Move 4: P_1 picks 7 and wins.

Now, let $E' = E \cup \{(2, 4, 6)\}$. We claim that the game played on $H' = (V, E')$ is a draw with optimal play.

In order to have any chance of winning, P_1 must pick vertex 1 in the first move. P_2 responds by picking 2.

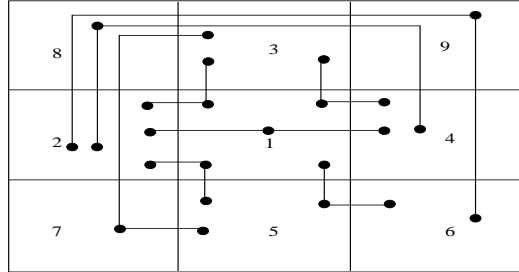
If P_1 picks 3 (respectively 5) in Move 2, P_2 picks 4 (respectively 6). P_1 is then forced to pick 6 (respectively 4) and P_2 picks 5 (respectively 3), forcing a draw.

If P_1 picks 4 (respectively 6) in Move 2, P_2 picks 3 (respectively 5), forcing a draw. Clearly, picking 7 in the second move makes a draw easier for P_2 .

Thus P_2 can always force a draw. Furthermore, it is clear that any such example requires at least seven vertices, since extra edges cannot prevent a three-move win.

Now we give an example of the induced extra set paradox for uniform hypergraphs.

Consider the hypergraph $H'' = (V'', E'')$ where $V'' = \{1, 2, \dots, 9\}$ and $E'' = \{(1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 3, 4), (1, 5, 6), (3, 5, 7), (2, 4, 8), (2, 6, 9)\}$.



As we have already seen, the first player can force a win on the subgraph induced by the vertex set $\{1, 2, \dots, 7\}$. We claim that the game on the entire hypergraph is a draw with optimal play.

In order to have any chance of winning, P_1 must pick vertex 1 in the first move. P_2 responds by picking 2.

If P_1 picks 3 (respectively 5) in Move 2, P_2 picks 4 (respectively 6). P_1 is then forced to pick 8 (respectively 9) and P_2 picks 5 (respectively 3), forcing a draw.

If P_1 picks 4 (respectively 6) in Move 2, P_2 picks 3 (respectively 5), forcing a draw. Clearly, picking 7, 8 or 9 in the second move makes a draw easier for P_2 .

We conclude that P_2 can always force a draw.