

# ON THE LARGEST $k$ -PRIMITIVE SUBSET OF $[1, n]$

**Sujith Vijay**

*Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, USA.*

sujith@math.rutgers.edu

*Received: 6/28/04, Revised: 1/24/06, Accepted: 1/27/06, Published: 2/1/06*

## Abstract

We derive bounds on the size of the largest subset of  $\{1, 2, \dots, n\}$  such that no element divides  $k$  others, for  $k \geq 3$  and sufficiently large  $n$ .

## 1. Introduction

Let  $S \subseteq \mathbb{N}$  be a finite set of positive integers. We say that  $S$  is  $k$ -primitive if no member of  $S$  divides  $k$  other elements in  $S$ .

Let  $f_k(n)$  denote the size of the largest  $k$ -primitive subset of  $[1, n]$ . It is well-known that  $f_1(n) = \lceil \frac{n}{2} \rceil$ . Lebensold [2] showed that, if  $n$  is sufficiently large,

$$(0.672\dots) < \frac{f_2(n)}{n} < (0.673\dots)$$

In this article, we show that, for  $k \geq 3$  and sufficiently large  $n$ ,

$$\frac{k}{k+1} + \frac{1}{8k^4} < \frac{f_k(n)}{n} < 1 - \frac{1}{8k \ln k}$$

Moreover, given  $\epsilon > 0$ , there exists  $k_0(\epsilon)$  such that for  $k \geq k_0(\epsilon)$  and  $n \geq n_0(k)$ ,

$$\frac{k}{k+1} + \frac{1-\epsilon}{k^4} < \frac{f_k(n)}{n} < 1 - \frac{1}{(2e^\gamma + \epsilon)k \ln k}$$

## 2. The Lower Bound

For  $\alpha \in \mathbb{R}$  and  $S \subseteq \mathbb{N}$ , we shall write  $\alpha S$  to denote the set  $\{\alpha x : x \in S\}$ . We begin by deriving a lower bound on  $f_k(n)$ .

Define  $S_0 = \{x : (k + 1)x > n\}$ , with  $|S_0| = \frac{nk}{k+1} + O(1)$ . Clearly,  $S_0$  is  $k$ -primitive. Let  $S_1 = \{x : \frac{n}{k+3} < x < \frac{nk}{(k+1)^2}, k(k+1) \mid x\}$ . Observe that any element in  $S_1$  has exactly  $k + 1$  other multiples in  $[1, n]$ . Let  $S_2 = (k + 1)S_1$ ,  $S_3 = (k + 2)S_1$  and  $S' = (S_0 \cup S_1) \setminus (S_2 \cup S_3)$ . Note that  $S'$  is  $k$ -primitive.

Let  $S_4 = (k + 1)^{-1}S_3$  and  $S_5 = k^{-1}S_2$ . Any element in  $S_4 \cup S_5$  has at most  $k$  other multiples in  $[1, n]$ . By construction, at least one of these will not occur in  $S'$ . Furthermore, no multiple of an element in  $S_4$ , except possibly itself, occurs in  $S_5$  and vice versa. It follows that  $S \doteq S' \cup S_4 \cup S_5$  is  $k$ -primitive.

Note that

$$|S_i| = \frac{n(k - 1)}{k(k + 1)^3(k + 3)} + O(1), \text{ for } 1 \leq i \leq 5$$

Furthermore,

$$S_i \cap S_j = \emptyset \text{ for } 1 \leq i < j \leq 5 \text{ except when } i = 4 \text{ and } j = 5.$$

Finally,

$$|S_4 \cap S_5| = \frac{n(k^3 - 4k - 1)}{k^2(k + 1)^5(k + 2)(k + 3)} + O(1)$$

Thus we have,

$$|S| = |S_0| + |S_1| - |S_4 \cap S_5| > n \left( \frac{k}{k + 1} + \frac{1}{8k^4} \right)$$

Note that for sufficiently large  $k$ ,

$$|S| > n \left( \frac{k}{k + 1} + \frac{1 - \epsilon}{k^4} \right)$$

### 3. The Upper Bound

Let  $S$  be a  $k$ -primitive subset of  $[1, n]$ . For a positive integer  $x \leq n/(k + 1)$ , let  $C_x \doteq \{x, 2x, \dots, (k + 1)x\}$  be the chain containing  $x$ . Observe that  $C_x \subseteq [1, n]$  and  $|S \cap C_x| \leq k$ . Thus if  $C_{x_1}, C_{x_2}, \dots, C_{x_m}$  are pairwise disjoint,  $|S| \leq n - m$ .

Let  $X = \{x : \frac{n}{2(k+1)} < x < \frac{n}{k+1}, x \text{ has no prime factor in } [2, k]\}$ . Thus if  $r \leq k$  and  $x \in X$ , we have  $(r, x) = 1$ .

We claim that  $\{C_{x_m}\}, x_m \in X$  is a pairwise disjoint collection.

Suppose not. Let  $rx_i = sx_j, x_i \neq x_j, 1 \leq r < s \leq k + 1$ . Since  $r \leq k$  and  $x_j \in X$ , we have  $(r, x_j) = 1$ . Thus  $x_j | x_i$ , i.e.,  $x_i \geq 2x_j > \frac{n}{k+1}$ , which is impossible. This proves our claim.

Let  $P_k$  denote the product of the prime numbers not exceeding  $k$ . The easy estimate  $P_k < 3^k$ , together with an application of the Chinese Remainder Theorem, yields

$$|X| = \frac{n}{2(k+1)} \prod_{p \leq k} \left(1 - \frac{1}{p}\right) + O(3^k)$$

By Mertens's theorem,

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \geq \frac{1}{e^{\gamma+\delta} \ln x} \text{ where } |\delta| < \frac{4}{\ln(x+1)} + \frac{1}{2x} + \frac{2}{x \ln x}$$

Computations for a bounded initial segment (suffices to consider  $x < 12000$ ) establish that

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \geq \frac{\ln 3}{3 \ln x} \text{ for } x \geq 3$$

Therefore, we obtain, for  $k \geq 3$ ,

$$|X| > \frac{n}{8k \ln k}$$

and for sufficiently large  $k$ ,

$$|X| > \frac{n}{(2e^\gamma + \epsilon)k \ln k}$$

## References

- [1] R. K. Guy, *Unsolved Problems in Number Theory*, Second Edition. Springer-Verlag, 1994.
- [2] K. Lebensold, *A divisibility problem*, Studies in Applied Mathematics 56, 1977.
- [3] F. Mertens, *Ein Beitrag zur analytischen Zahlentheorie*, Journal für die Reine und Angewandte Mathematik 78, 1874.