# On the Discrepancy of Staircase Sequences

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Combinatorial discrepancy is a hypergraph parameter that has been studied extensively for several special families, such as arithmetic progressions and rows of Hadamard matrices. A well-known random-colouring argument yields an upper bound of $O(n^{1/2+\epsilon})$ for any hypergraph on $n$ vertices where the number of edges is bounded by a polynomial in $n$. At the other end of the spectrum, we have the complete hypergraph with $2^n$ edges and discrepancy $n/2$. Probabilistic considerations show that there exists a family of hypergraphs $\{H_k^*\}$ with fewer than $(1+(1/k))^n$ edges and discrepancy at least $c_0 n/\sqrt{k}$, for some absolute constant $c_0$. The purpose of this note is to give an explicit example of a family of hypergraphs $\{H_k\}$with fewer than $(1+(1/k))^n$ edges and discrepancy at least $n/(2k^2+o(k^2))$.

The hypergraphs are defined on the vertex $V = \{0, 1, \ldots, n-1\}$. Let $0 \leq a_0 < a_1 < \cdots < a_m < n$. Then the edge $\{a_0, a_1, \ldots, a_m\}$ belongs to $H_k$ if and only if $a_j - a_{j-1} = k$ or $k+1$ for all $j, 1 \leq j \leq m$. We shall refer to $H_k$ as the hypergraph of *staircase sequences of stepsize* $k$. It is easy to see that $H_k$ has asymptotically $c^n$ edges, where $c$ is the unique positive solution to the equation $c^{k+1} = c+1$. Let $\delta_k = 1/(k+1)$. Note that by intermediate value theorem, $c \in (1, 1+\delta_k)$ for $k \geq 2$. We will now show that $H_k$ has discrepancy at least $n/(2k^2+o(k^2))$.

Let $\chi : V \to \{+1, -1\}$ be any 2-colouring of $V$, and let $M$ be the discrepancy under this colouring. Assume without loss of generality that $|\{j : \chi(j) = 1\}| \geq |\{j : \chi(j) = -1\}|$. We define $\chi^* : V \to \{0, 1, 2, \ldots, k\}$ as follows:

- $\chi^*(0) = (1 - \chi(0))/2$
- $0 \leq \chi^*(n) < k \Rightarrow \chi^*(n+1) = \chi^*(n) + 1$

- $\chi^*(n) = k$ and $\chi(n+1) = -1 \Rightarrow \chi^*(n+1) = 0$
- $\chi^*(n) = k$ and $\chi(n+1) = 1 \Rightarrow \chi^*(n+1) = 1$

Let $p_j = |\{i : \chi(i) = 1 \text{ and } \chi^*(i) = j\}|$, and let $m_j = |\{i : \chi(i) = -1 \text{ and } \chi^*(i) = j\}|$. Observe that $\{\chi^*(i) = j\}$ forms a staircase sequence of stepsize $k$, for $1 \leq j \leq k$. Note that $p_0 = 0$ and $m_0 \leq \sum_{j=1}^{k}(p_j - m_j) \leq kM$.

Observe that $\chi^*(j) = 1$ and $\chi(j) = -1$ imply that $\chi^*(j-1) = 0$ and $\chi(j-1) = -1$. Thus $m_1 \leq m_0 + 1 \leq kM + 1$ and $b_1 \leq m_1 + M \leq (k+1)M + 1$. But $m_1 + b_1 \geq n/(k+1)$. It follows that

$$M \geq \frac{n}{(k+1)(2k+1)} - \frac{2}{2k+1} = \frac{n}{2k^2 + o(k^2)}$$