Is $f(x+y)$ the same as $f(x)+f(y)$?

Depends on the function f, certainly not always. **If a continuous function f satisfies $f(x+y) = f(x) + f(y)$, then its graph must be a straight line passing through the origin. Such functions are said to be linear.**

It is easy to see that $f(x) = ax$ is linear. After all, $f(x+y) = a(x+y) = ax+ay = f(x)+f(y)$. But why are these the only continuous linear functions?

Suppose you have a function f with $f(x+y) = f(x) + f(y)$ for all x, y. Put $x = y = 0$ and you get $f(0) = 0$. Let $f(1) = a$. Put $x = y = 1$ and you get $f(2) = 2f(1) = 2a$. Put $x = 1, y = 2$ and you get $f(3) = 3a$. Put $x = y = \frac{1}{2}$ and you get $f(\frac{1}{2}) = \frac{1}{2}a$. It is a little harder to prove that $f(\pi) = \pi a$, and that is where you use continuity.[1]

Anyway, the point is that **most functions are not linear**. Consider the following functions:

$$g_1(x) = x^2 \qquad g_2(x) = \sin x \qquad g_3(x) = e^x$$
$$g_4(x) = \ln x \qquad g_5(x) = \sqrt{x} \qquad g_6(x) = \frac{1}{x}$$

The first three are continuous everywhere. The last three are continuous wherever they are defined. None of them are linear.

It is important to know when a function is not linear. Since none of the above functions are linear, none of the following statements are correct.

- $(x+y)^2 = x^2 + y^2$ (WRONG, even when $x = y$)
- $\sin(x+y) = \sin x + \sin y$ (WRONG, even when $x = y$)
- $e^{x+y} = e^x + e^y$ (WRONG, even when $x = y$)
- $\ln(x+y) = \ln x + \ln y$ (WRONG, even when $x = y$)
- $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ (WRONG, even when $x = y$)
- $\frac{1}{x+y} = \frac{1}{x} + \frac{1}{y}$ (WRONG, even when $x = y$)

[1] If your head hurts, you can always believe me.