16 Extremal Combinatorics

Definitions

- A p-clique in a graph G is a complete subgraph induced on p vertices of G.
- A collection of subsets is said to be an antichain if no subset is contained in another.

Theorems

- Let G be a graph which does not have any p-cliques. Turan's theorem states that the number of edges in G is at most $\left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}$
- Let $\mathcal{F}$ denote the family of subsets of $\{1, 2, \ldots, n\}$. Sperner's theorem states that the largest antichain in $\mathcal{F}$ has size $\binom{n}{\lfloor n/2 \rfloor}$

17 Ramsey Theory

Basics

- $R(m, n)$ is the smallest positive integer with the property that no matter how the edges of the complete graph on $R(m, n)$ vertices are coloured red or blue, there exists a red m-clique or a blue n-clique.
- $R(m, n) \leq R(m, n-1) + R(m-1, n)$
- $W(k)$ is the smallest positive integer with the property that no matter how the first $W(k)$ positive integers are 2-coloured, there exists a monochromatic arithmetic progression of k terms.

Results

- In any group of six people, there are three mutual friends or three mutual strangers. But there exist groups of five people without three mutual friends or three mutual strangers. Thus $R(3, 3) = 6$
- $R(m, n) \leq \binom{m+n-2}{m-1}$
- $R(n, n) < 4^{n-1}$
- $W(3) = 9$.
- Any sequence of $mn + 1$ distinct real numbers has an increasing subsequence of length $m + 1$ or a decreasing subsequence of length $n + 1$.

18 The Probabilistic Method

Theory

- If X is a non-negative integer-valued random variable with $E(X) < 1$, there exists an instance where X takes the value 0.

- If X is any random variable, there exists an instance where the value of X less than or equal to $E(X)$ and also an instance where the value of X is greater than or equal to $E(X)$.

Practice

- $R(n, n) \geq 2^{n/2}$

- There exists a tournament on n vertices with at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.

⋆ A random permutation of vertices corresponds to a Hamiltonian path with probability $\frac{1}{2^{n-1}}$. Since there are $n!$ permutations of n vertices, it follows from the linearity of expectation that the average number of Hamiltonian paths in a random tournament is $\frac{n!}{2^{n-1}}$. Thus there exists a tournament on n vertices with at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.

- $W(k) \geq 2^{k/2}\sqrt{(k-1)/2}$

19 Combinatorial Games

Basics

- In n^d tic-tac-toe, players alternately mark previously unmarked cells with distinguishing marks, and the winner is the first to get n-in-a-line.
- A pairing strategy in tic-tac-toe is a pairing of cells on the board such that each winning line passes through some pair.
- Nim is played on an initial configuration of two or more piles of stones. In each turn, players take away one or more stones from a single non-empty pile. The player who makes the last move wins.
- Chomp is played on an initial configuration of a rectangular $m \times n$ grid. In each turn, players remove a square and all squares above and to the right of it. The player to take the lower-left-hand square loses.

Results

- In any game where extra moves don't hurt, the second player cannot have a winning strategy.
- The number of winning lines in n^d tic-tac-toe is $\frac{(n+2)^d - n^d}{2}$.
- For pairing strategy to work in tic-tac-toe, the number of cells must be at least as large as twice the number of winning lines.
- The best option in Nim is to divide the number of stones in each pile into subpiles corresponding to distinct powers of 2 and take away stones, if possible, leaving an even number of subpiles for each power of 2.
- In Chomp, the first player has a winning strategy.