A game based on a question of Erdős

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1 Introduction

Let $S_N \doteq \{0, 1, 2, \ldots, N\}$. A classic result in discrepancy theory is the theorem of Roth [1] stating that if the elements of S_N are 2-coloured, there exists an arithmetic progression of discrepancy at least $\frac{1}{20}N^{1/4}$. Matoušek and Spencer [7], building upon results by Sárközy (see [2]) and Beck [4], showed that apart from constants, this result is best possible. The parallel question for the subfamily of arithmetic progressions containing 0, raised by Erdős in the 1930s (see [3]), remains open. An upper bound of $O(\log N)$ follows from a multiplicative colouring based on congruence classes modulo 3, but it is not even clear whether the discrepancy is unbounded.

2 A Maker-Breaker Game

We consider a family of two-player games where the players, Maker and Breaker, alternately colour (uncoloured) the elements of S_N red and blue until all elements are coloured. Maker wins if (s)he has a lead of ℓ on some arithmetic progression containing 0 and Breaker wins otherwise.

For every $\epsilon > 0$, we exhibit an explicit winning strategy for Maker (respectively Breaker) if $\ell < N^{\frac{1}{2}-\epsilon}$ (respectively $\ell > N^{\frac{1}{2}+\epsilon}$), for sufficiently large N. The proofs make use of potential functions introduced by Szekely [5] and Beck [6].

Let N be sufficiently large, and let P_N denote the set of primes not exceeding N. Let $k = \lceil \frac{1}{2\epsilon} \rceil$. Define $P^* = \{p \in P_N : N/2 < p^k < N\}$ and let U denote the set of all k-fold products of distinct elements in P^*. Let $|P^*| = m$ and $|U| = n \doteq \binom{m}{k}$. Observe that for every $Q \subseteq P^*$ with $|Q| = k$, there is a unique integer $q \in [1, N]$ such that q is divisible by all the elements in Q. Furthermore, this integer q belongs to U.

Maker shall secure the required lead on an arithmetic progression of the form $A_p \doteq \{0, p, 2p, \ldots\} \cap S_N$ where $p \in P^*$. Let $\mathcal{F}$ denote the family of such arithmetic progressions. We associate a counter c_i with each element $A_{p_i} \in \mathcal{F}$. The counter C_i is initialised to 0, and incremented (respectively decremented) by 1 every time Maker (respectively Breaker) chooses a multiple of p_i. We shall denote the value of C_i at the end of Breaker's t^{th} move by $c_{i,t}$.

Maker begins by choosing 0. For all further moves, Maker will choose uncoloured elements of U. Suppose the elements $0, v_1, u_2, v_2, \ldots, u_t, v_t$ were chosen already and it is Maker's turn. Note that

$$\sum_{i=1}^{m} c_{i,t} \geq \sum_{i=1}^{m} c_{i,1} \geq m - k > 0$$

For $u \in U$, define $f_t(u) = f_t(p_{i_1} \cdots p_{i_k}) = c_{i_1,t} + \cdots + c_{i_k,t}$. Maker chooses the element u_t^* that maximises the value of f_t over all uncoloured elements of U. Let $m^* \doteq \sqrt{\frac{m^{k-1}}{5^k k!}}$ and $x^+ \doteq \max(x, 0)$. Consider the potential function

$$V_t = \sum_{i=1}^{m} [(c_{i,t} + m^*)^+]^2$$

A straightforward computation shows that irrespective of Breaker's reply, the potential function will increase by at least 2, provided the counters associated with u_t^* satisfy $c_{i_r,t} \geq -m^* \ \forall\, r \in \{1, 2, \ldots, k\}$. We claim that Maker can find u_t^* with this property for the first $n_0 \doteq \binom{m/3}{k}$ moves.

Call a progression A_{p_i} *unbreakable* if $c_{i,t} \geq \frac{m^*}{2k}$. Note that if we have an unbreakable progression, then we are done immediately, since any lead can be sustained.

If there are no unbreakable progressions, the family $\mathcal{F}'$ of progressions A_{p_j} satisfying $c_{j,t} > -\frac{m^*}{2k}$ has at least $\frac{m}{2}$ elements, and the claim will follow if we can show that $f_t(u_t^*) > -\frac{m^*}{2}$. Note that for the first n_0 moves, there exist distinct progressions $A_{p_{j_1}}, \ldots, A_{p_{j_k}} \in \mathcal{F}'$ such that $u_t' \doteq p_{j_1} \cdots p_{j_k}$ is not coloured and $c_{j_r,t} > -\frac{m^*}{2k}$ $\forall\, r \in \{1, 2, \ldots, k\}$, yielding $f_t(u_t^*) \geq f_t(u_t') > -\frac{m^*}{2}$, as required.

Thus, at the end of n_0 moves, the potential reaches a value $V_{t_0} > n_0 > \frac{m^k}{4^k k!}$ Therefore, there exists i_0 satisfying

$$c_{i_0,t_0} > m^* > cN^{\frac{1}{2}-\epsilon} > \ell$$

Having established the required lead on $A_{p_{i_0}}$, Maker picks uncoloured elements from this progression and retains the advantage till the end.

Now let $\{P_i\}$ be an enumeration of the arithmetic progressions in $\{0, 1, \ldots, N\}$, containing 0. Suppose $x_1, y_1, x_2, y_2, \ldots, x_k$ are already chosen and it is Breaker's turn. Let m_i and b_i denote the number of elements in P_i that are chosen by Maker and Breaker respectively. Let

$$w_k(P_i) \doteq (1 + \frac{1}{\sqrt{N}})^{m_i}(1 - \frac{1}{\sqrt{N}})^{b_i} \text{ and } T_n \doteq \sum_i w_k(P_i)$$

The weight of $x \in X$ is given by

$$w_k(x) = \sum_{x \in P_i} w_k(P_i)$$

Breaker chooses y_k of maximal weight among the uncoloured elements of X. Note that $T_1 = |\{P_i\}| << N^2 \ln N$. Furthermore,

$$T_{k+1} \leq T_k - \frac{w_k(y_k) - w_k(x_{k+1})}{\sqrt{N}} \leq T_k$$

Suppose $\exists\, i$ such that $r_i - m_i > 3\sqrt{N} \ln N$. Since $(1 - \frac{1}{N})^{b_i} > e^{-1}$, we get $w_k(P_i) >> n^3$, a contradiction. It follows that Breaker can always prevent a lead of $N^{\frac{1}{2}+\epsilon}$, for sufficiently large N.

References

1. K. F. Roth, *Remark concerning integer sequences.* Acta Arithmetica 9, 1964.

2. P. Erdős and J. Spencer, *Probabilistic Methods in Combinatorics.* Akadémiai Kiadó, Budapest, 1974.

3. P. Erdős, *On the combinatorial problems which I would most like to see solved.* Combinatorica 1, 1981.

4. J. Beck, *Roth's estimate of the discrepancy of integer sequences is nearly sharp.* Combinatorica 1, 1981.

5. L. A. Szekely, *On two concepts of discrepancy in a class of combinatorial games*, In Finite and Infinite Sets, Colloq. Math. Soc. Janos Bolyai, Vol.37. North-Holland, 1984.

6. J. Beck, *Deterministic graph games and a probabilistic intuition.* Combinatorics, Probability and Computing 3, 1994.

7. J. Matoušek and J. Spencer, *Discrepancy in arithmetic progressions.* Journal of the American Mathematical Society 9, 1996.