On the Ultimate Periodicity of Mex Sequences

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Suppose we are given a finite sequence of non-negative integers $t_1, t_2, \ldots, t_k$. We extend this to an infinite sequence as follows. For all $n \geq k$, we define

$$t_{n+1} = mex(t_1 + t_n, t_2 + t_{n-1}, \ldots, t_n + t_1)$$

where $mex(a_1, \ldots, a_n)$ denotes the minimum excludent, or the least non-negative integer not appearing in the sequence $a_1, a_2, \ldots, a_n$.

In [1,**E27**], it is asked if these mex sequences are ultimately periodic. The motivation for studying these sequences comes from the analysis of octal games using Sprague Grundy Theory, where ordinary addition is replaced by nim-addition.

We restrict ourselves to input sequences with all terms positive, and propose the following:

Conjecture 1 The extended sequence is periodic with period $3k + 2$, where k is the length of the input sequence.

Clearly, the truth of the above conjecture will imply that all such sequences are bounded. We conjecture that the upper bound depends only on the length of the input sequence, and not on the terms themselves.

Conjecture 2 The extension of any input sequence of length k has upper bound $2k$.

We begin by proving the following lemma, which shows that appearances are not always deceptive.

Lemma 1 Let $t_1, t_2, \ldots, t_k$ be the given input sequence and let t_n denote the n^{th} term in the extended sequence. Suppose there exist integers ℓ and $m, \ell \leq m$, such that the first $2(m+\ell)$ terms of the sequence are

$$a_1, a_2, \ldots, a_\ell, b_1, b_2, \ldots, b_m, b_1, b_2, \ldots b_m, b_1, b_2, \ldots, b_\ell$$

Then $t_n = t_{n-m}$ for all $n > m + \ell$.

Proof We proceed by induction on j, and it will be clear from the induction step that the base case holds.

Let $n = 2m + \ell + j, j \geq \ell + 1$. Observe that if $s \leq \lfloor \frac{n}{2} \rfloor < 2m + j$, then $n - s > m + \ell$, so that $t_{n-s} = t_{n-s-m}$.

Thus $t_n = mex(\{t_s + t_{n-s} : 1 \leq s \leq \lfloor \frac{n}{2} \rfloor\}) = mex(\{t_s + t_{n-m-s} : 1 \leq s \leq \lfloor \frac{n}{2} \rfloor\} = t_{n-m}$. This completes the proof. ▌

We now prove Conjecture 1 for a special case.

Theorem 1 Suppose all terms in the input sequence are greater than 1. Then the extended sequence is periodic with period $3k + 2$.

Proof Note that the extended sequence has the following form:

$$t_{1:k}\ 0^{k+1}\ 1^{k+1}\ 2^k\ 0^{k+1}\ 1^{k+1}\ 2^k\ 0^{k+1} \cdots$$

Therefore, by Lemma 1, the sequence is periodic with period $3k+2$. ▌

Let $t_{1:k}$ be the input sequence, with $t_r = 1$, and $t_s > 1$ for $r + 1 \leq s \leq k$

The extended sequence has the following form:

$$t_{1:k}\ 0^{k+1}\ A^1_{1:r}\ 1^{k+1}\ B^1_{1:k-r}\ 0^{k+1}\ A^2_{1:r}\ 1^{k+1}\ B^2_{1:k-r}\ 0^{k+1}\ A^3_{1:r} \cdots$$

Conjecture 3 For $p \geq 1$,

$$A^p_j \leq A^{p+1}_j, 1 \leq j \leq r \text{ and } B^p_j \leq B^{p+1}_j, 1 \leq j \leq k - r$$

Note that Conjecture 3, together with boundedness of the extended sequence (weaker than Conjecture 2) implies Conjecture 1.

Let $\mathbf{A}^p$ and $\mathbf{B}^p$ abbreviate $A^p_{1:r}$ and $B^p_{1:k-r}$, respectively.

Example Let the input sequence be 5,4,2,1,2,3. Then $\mathbf{A}^1 = 6, 5, 4, 4$; $\mathbf{A}^2 = 9, 8, 4, 4$; $\mathbf{A}^p = 11, 8, 4, 4$ for $p \geq 3$ and $\mathbf{B}^p = 7, 7$ for $p \geq 1$.

On a personal note, the above example was as instructive as it was elusive - ending, rather anti-climactically, a week-long effort to prove that $\mathbf{A}^2 = \mathbf{A}^3$. Note that in view of Lemma 1, $\mathbf{A}^2 = \mathbf{A}^3 = \mathbf{A}^4$ together with $\mathbf{B}^1 = \mathbf{B}^2 = \mathbf{B}^3$ would have settled Conjecture 1 in the affirmative.

Lemma 2 $\mathbf{A}^1$ is non-increasing.

Proof We shall prove that $A^1_j \geq A^1_{j+1}$ for $1 \leq j \leq r-1$. We proceed by induction on j.

For $j \geq 2$, we define $T^1_j = \{A^1_{1:j-1} + t_{j-1:1}, t_{j:k}, 0\}$, so that $A^1_j = mex(T^1_j)$

Note that $A^1_2 \leq A^1_1$, since $A^1_1 \notin T_2$.

Now assume that $A^1_1 \geq A^1_2 \geq \cdots \geq A^1_\ell$

Observe that $A^1_\ell \notin T^1_{\ell+1}$. Thus $A^1_{\ell+1} \leq A^1_\ell$. This completes the proof. $\blacksquare$

Exactly along the same lines, it can be proved that $\mathbf{B}^1$ is non-increasing.

Thus $A^1_j = mex(t_{j:k}, 0)$ and $B^1_j = mex(0, 1, t_{j:k} + 1, \mathbf{A}^1)$. It follows that $\mathbf{A}^1 \cap \mathbf{B}^1 = \emptyset$

For $1 \leq s \leq r$, let $\widetilde{s} \doteq j + k - r$

Theorem 2 $A^1_j \leq A^2_j, 1 \leq j \leq r$

Proof Let $T^2_j = \{t_{1:j-1} + A^2_{j-1:1}, t_{j:k}, 0, \mathbf{B}^1, A_{1:\widetilde{j}}, A_{\widetilde{j}+1:r} + A_{r:\widetilde{j}+1}\}$, so that $A^j_2 = mex(T^2_j)$.

Since $A^1_j = mex(t_{j:k}, 0)$, it follows that $\{0, 1, \ldots, A_j - 1\} \subseteq \{0, t_{j:k}\} \subseteq T_j$. Thus $A^1_j \leq A^2_j$ $\blacksquare$

Lemma 3 The equation $x - y = 1$ has no solutions in $\mathbf{B}^1$.

Proof Since $\mathbf{B}^1$ is non-increasing, it suffices to prove that $B^1_{j-1} \neq B^1_j - 1$ for all j.

Let $J = \{j : A^1_{j-1} \geq A^1_j + 2\}$. Let $j_1, \ldots, j_m$ be the elements of J in increasing order. Define $j_0 = 1$ and $j_{m+1} = r$.

Let $2 \leq s \leq r$, and let i be such that $j_i < s \leq j_{i+1}$.

We have, $B^1_s = mex(t_{s;k} + 1, 1, \mathbf{A}^1, 0)$

Clearly, $A^1_{j_i} + 1 \notin \mathbf{A}^1$. Since $j_i < s$, $A^1_{j_i} \neq t_\ell, s \leq \ell \leq k$. Thus $B^1_s \leq A^1_{j_i} + 1$.

Let $q = j_{i+1}$. Since $A^1_q = mex(0, t_{q:k})$ and $s \leq q$, it follows that

$$\{0, 1, \ldots, A^1_q - 1\} \subseteq \{t_s, t_{s+1}, \ldots, t_k\}$$

Furthermore, $A^1_q, A^1_q + 1, \ldots, A^1_{j_i} \in \mathbf{A}^1$. Thus $B^1_s = A^1_{j_i} + 1$.

Similarly, if $s > r$, it can be shown that $B^1_s = A^1_{j_m} + 1$.

Observe that the successive elements of $\{A^1_{j_i}\}, 1 \leq i \leq m$ differ by at least 2. Thus $x - y = 1$ has no solutions in $\mathbf{B}^1$. ∎

Theorem 3 $\mathbf{B}^1 = \mathbf{B}^2$

Proof We proceed by induction on j. As in the proof of Theorem 1, we do not deal with the base case separately.

Let $S^1_j = \{t_{1:j-1} + B^1_{j-1:1}, t_{j;k} + 1, 1, \mathbf{A}^1, 0\}$ so that $B_j = mex(S_j)$

Case 1: $1 \leq j \leq r$.

We have, $B^2_j = mex(t_{1:j-1} + B^2_{j-1:1}, t_{j:k} + 1, 1, \mathbf{A}^2, 0, \mathbf{A}^1, 1, B^1_{k-r:1} + 1, 2)$

By induction hypothesis, $B^2_j = mex(S^1_j, \mathbf{A}^2, B^1_{k-r:1} + 1)$

Note that $A^2_s = mex(t_{1:s-1} + A^2_{s-1:1}, t_{s;k}, 0, \mathbf{B}^1, 1, A_{1:\widetilde{s}}, A_{\widetilde{s}+1:r} + A_{r:\widetilde{s}+1})$. Thus $\mathbf{B}^1 \cap \mathbf{A}^2 = \emptyset$. Also, by Lemma 3, $x = y + 1$ has no solutions in $\mathbf{B}^1$. It follows that $B^1_j = B^2_j$

Case 2: $r + 1 \leq j \leq k$

Let $p = j - r$. We have,

$$B^2_j = mex\left(t_{1:j-1} + B^2_{j-1:1}, t_{j:k} + 1, 1, \mathbf{A}^2, 0, \mathbf{A}^1, 1, B^1_{k-r:p} + 1, B^1_{1:p} + B^1_{p:1}\right)$$

By induction hypothesis, $B^2_j = mex\left(S^1_j, \mathbf{A}^2, B^1_{k-r:p} + 1, B^1_{1:p} + B^1_{p:1}\right)$

As in Case 1, $B^1_j \notin \mathbf{A}^2$ and $B^1_j \neq B^1_s + 1, p \leq s \leq k - r$. We also have $B^1_i + B^1_{p+1-i} > B^1_j$ for $1 \leq i \leq p$. Therefore, $B^1_j = B^2_j$. ▮

Reference

1. Richard K. Guy, *Unsolved Problems in Number Theory*, Springer-Verlag, 1994.