

A Tale of Two Square Roots

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Given $k + 1$ positive integers a_1, \dots, a_k and M , it is not known whether

$$\sum_{i=1}^k \sqrt{a_i} \leq M$$

is decidable in polynomial time. Decision problems of this kind arise, for example, in the computation of short polygonal paths connecting vertices with integer coordinates. The difficulty lies in estimating the precision to which computations must be carried out. This depends on the smallest non-zero fractional part of the sum of square roots of k integers less than or equal to N . Here we obtain an asymptotic estimate for the case $k = 2$.

For a positive integer N , we define

$$d(x, y) \doteq \|\sqrt{x} + \sqrt{y}\|$$

where $\|\alpha\|$ denotes the distance from α to the integer nearest to α . Let $S_2(N) = \{d(x, y) : 0 \leq x, y \leq N\}$, and let $d_2(N)$ denote the least positive element in $S_2(N)$. In [1], it is shown that $d_2(N) = \Theta(N^{-3/2})$. The purpose of this note is to prove that

$$d_2(N) \sim \frac{1}{16N^{3/2}}$$

Let $x, y \leq N$. Interchanging x and y if necessary, write $x = A^2 + a$ with $-A \leq a \leq 0$ and $y = B^2 + b$ with $0 \leq b \leq B$, where A and B are integers. Observe that $p \doteq A + B$ is the integer nearest to $\sqrt{x} + \sqrt{y}$. It can be verified that $p \leq 2\sqrt{N}$. Therefore,

$$\begin{aligned}
d(x, y) &= |(\sqrt{x} + \sqrt{y}) - p| \\
&= \left| \frac{(\sqrt{x} + \sqrt{y})^2 - p^2}{(\sqrt{x} + \sqrt{y}) + p} \right| \\
&= \left| \frac{[(x + y - p^2) + 2\sqrt{xy}] [(x + y - p^2) - 2\sqrt{xy}]}{(\sqrt{x} + \sqrt{y} + p) [(x + y - p^2) - 2\sqrt{xy}]} \right| \\
&= \left| \frac{(x + y - p^2)^2 - 4xy}{(\sqrt{x} + \sqrt{y} + p) [(\sqrt{x} - \sqrt{y})^2 - p^2]} \right| \\
&\geq \frac{1}{(4\sqrt{N})(4N)} \\
&= \frac{1}{16N^{3/2}}
\end{aligned}$$

Observe that a matching upper bound will follow if we show that for $\epsilon > 0$, and all $N \geq N_0(\epsilon)$, there exist $x, y \in [(1 - \epsilon)N, N]$ and $p = \lfloor \sqrt{x} + \sqrt{y} + 1/2 \rfloor$ satisfying $(x + y - p^2)^2 - 4xy = 1$. In terms of a, b, A and B , the equation becomes

$$(a - b)^2 - 4(A + B)(Ab + Ba) = 1$$

Out of kindness to the reader, we shall cut a long story short, and assert that

$$a = -4t^2, b = 4t^2 + 1, A = 8t^3 + t, B = 8t^3 + 3t$$

satisfy the above equation for all values of t . Thus

$$x = 64t^6 + 16t^4 - 3t^2 \doteq f(t), y = 64t^6 + 48t^4 + 13t^2 + 1 \doteq g(t), p = 16t^3 + 4t \doteq h(t)$$

is an infinite 1-parameter family of solutions to $(x + y - p^2)^2 - 4xy = 1$, with $p = \lfloor \sqrt{x} + \sqrt{y} + 1/2 \rfloor$.

Given $\epsilon > 0$, let m_0 be the least integer such that

$$\frac{g(m+1)}{f(m)} < \frac{1}{1-\epsilon} \quad \forall m \geq m_0$$

Let $N_0 = g(m_0)$. For $N \geq N_0$, let ℓ be the largest integer such that $g(\ell) \leq N$. Clearly, $(1 - \epsilon)N \leq f(\ell) < g(\ell) \leq N$. Moreover, $x = f(\ell)$, $y = g(\ell)$, $p = h(\ell)$ satisfy $(x + y - p^2)^2 - 4xy = 1$. This completes the proof.

References

1. Dana Angluin and Sarah Eisenstat, *How close can $\sqrt{a} + \sqrt{b}$ be to an integer?*, Technical Report YALEU/DCS/TR-1279, Department of Computer Science, Yale University, February 2004.

<ftp://ftp.cs.yale.edu/pub/TR/tr1279.pdf>

2. Erik D. Demaine, Joseph S. B. Mitchell and Joseph O'Rourke, *The Open Problems Project*.

<http://maven.smith.edu/~orourke/TOPP/P33.html>