

Which Two-Sorted Algebras of Booleans and Naturals have a Finite Basis?

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ABSTRACT. We show that the two-sorted algebra of Booleans and naturals with conjunction, addition and inequality is not finitely based. If addition is removed, or negation is included, then the resulting algebra is finitely based.

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1. Introduction

A basis for an algebra A is a set of equations E such that an equation is true in A if and only if it is a logical consequence of E . The existence of finite bases for algebras is a classic topic of study in universal algebra (see, e.g., [17]), dating back to Lyndon [14]. Murskiĭ [19] proved that “almost all” finite algebras (namely all quasi-primal ones) are finitely based, while in [18] he presented an example of a three-element algebra that has no finite basis. McKenzie [16] settled Tarski’s Finite Basis Problem in the negative, by showing that the question whether a finite algebra is finitely based is algorithmically undecidable.

As an example of equational logic, Henkin [13] showed that the algebra of natural numbers with addition and multiplication is finitely based. (Subsequently we will refer to the algebra of natural numbers as the algebra of naturals.) Gurevič [12] showed that the algebra of naturals with addition, multiplication and exponentiation is not finitely based. The proof is based on an identity by Wilkie [22], which provided a negative answer to Tarski’s High School Algebra Problem. Aceto, Ésik and Ingólfssdóttir [1] proved that a set of equations in finitely many variables (so in particular a finite set of equations) cannot be a basis for the algebra of naturals with addition and maximum.

In this paper we consider two-sorted algebras of Booleans and naturals. We start with proving that the algebra with successor, conjunction and inequality is finitely based. We extend the algebra with addition and present an infinite set of equations, which we prove to be a basis for this second algebra. The proof relies heavily on results for so-called cutting planes [11] by Chvátal [7] and Schrijver [20] (see also [21, Chapter 23]). Furthermore, we show that the second algebra is not finitely based. Next, we extend the second algebra with negation and prove that the resulting third algebra is finitely based. Finally we extend the third algebra with a conditional operator, so that it encompasses the maximum operator, and prove that, in contrast with the negative result obtained in [1], this fourth algebra is finitely based.

As further related work, iterative algebras without a finite basis can be found in [2, 8, 9], and process algebras without a finite basis in [3, 10].

2. Inequality and conjunction

We start our study of two-sorted algebras of Booleans $\mathbb{B} = \{T, F\}$ and naturals \mathbb{N} with the algebra $\mathcal{A} = \langle \mathbb{B}, \mathbb{N}; T, F, 0, S, \wedge, \preceq \rangle$. Here, $S : \mathbb{N} \rightarrow \mathbb{N}$ denotes the successor, $\wedge : \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B}$ is conjunction and $\preceq : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{B}$ represents the smaller-or-equal relation. Note that T is definable as $0 \preceq 0$. We assume two countably infinite sets of variables of sorts Boolean and natural. A valuation is a sort preserving mapping from variables (of sort Boolean and natural) to \mathbb{B} and \mathbb{N} . As notational binding convention, S binds strongest and \wedge binds weakest. Let $S^0(t)$ denote t and let $S^{n+1}(t)$ denote $S(S^n(t))$.

The axioms (A1-11) below are all true in \mathcal{A} . Let Σ denote the equational base consisting of these axioms. Let u, v, w denote variables of sort Boolean and x, y, z variables of sort natural. In this section we show that Σ is a finite basis for \mathcal{A} . In other words, given two terms s and t over this algebra such that $\sigma(s)$ coincides with $\sigma(t)$ for all valuations σ , it can be proved using Σ that $s \approx t$ with the usual rules of equational logic (see, e.g., [13]).

$$\begin{aligned}
\text{(A1)} \quad & u \wedge T \approx u \\
\text{(A2)} \quad & u \wedge F \approx F \\
\text{(A3)} \quad & u \wedge v \approx v \wedge u \\
\text{(A4)} \quad & u \wedge (v \wedge w) \approx (u \wedge v) \wedge w \\
\text{(A5)} \quad & u \wedge u \approx u \\
\text{(A6)} \quad & 0 \preceq x \approx T \\
\text{(A7)} \quad & x \preceq x \approx T \\
\text{(A8)} \quad & x \preceq S(x) \approx T \\
\text{(A9)} \quad & S(x) \preceq x \approx F \\
\text{(A10)} \quad & S(x) \preceq S(y) \approx x \preceq y \\
\text{(A11)} \quad & (x \preceq y \wedge y \preceq z) \wedge x \preceq z \approx x \preceq y \wedge y \preceq z
\end{aligned}$$

In view of (A3,4), Boolean terms can be taken modulo associativity and commutativity of conjunction. Let $s \approx_{\Sigma} t$ denote that the equation can be derived from Σ . By (A10) we have $S^n(x) \preceq S^n(y) \approx_{\Sigma} x \preceq y$ for all $n \geq 0$. We say that a

Boolean term b is *satisfiable* if $\sigma(b) = T$ for some valuation σ ; in this case, σ is said to *satisfy* b .

Let b and c be Boolean terms, not containing any Boolean variables, that coincide under all valuations. We develop the machinery to prove $b \approx_\Sigma c$.

Let V_{bc} be a non-empty finite set of variables of sort natural that includes all variables occurring in b or c . Consider a Boolean term d that only contains variables from V_{bc} . Using (A7,9) to rewrite conjuncts T and F , d takes the form

$$s_1 \preceq t_1 \wedge \cdots \wedge s_n \preceq t_n$$

where the s_i and t_j are of the form $S^m(0)$ or $S^m(x)$ with $x \in V_{bc}$. In view of (A6), we can add conjuncts $0 \preceq x$ for all $x \in V_{bc}$; let \hat{d} denote the resulting Boolean term.

We construct a weighted directed graph G_d . Its vertex set P is $V_{bc} \cup \{0\}$. Furthermore, each conjunct in \hat{d} , say $S^m(p) \preceq S^n(q)$ with $p, q \in P$, gives rise to an edge from p to q with cost $n - m$. The cost of a path in G_d is the sum of the costs of its edges.

Lemma 1. $S^n(x) \preceq x \approx_\Sigma F$ for $n \geq 1$.

Proof. By induction on n . The base case $n = 1$ is (A9). The inductive case goes as follows:

$$\begin{aligned} & S^{n+1}(x) \preceq x \\ \approx_\Sigma & S^{n+1}(x) \preceq x \wedge x \preceq S(x) & \text{(A1, 8)} \\ \approx_\Sigma & S^{n+1}(x) \preceq x \wedge x \preceq S(x) \wedge S^{n+1}(x) \preceq S(x) & \text{(A11)} \\ \approx_\Sigma & S^{n+1}(x) \preceq x \wedge x \preceq S(x) \wedge S^n(x) \preceq x & \text{(A10)} \\ \approx_\Sigma & S^{n+1}(x) \preceq x \wedge x \preceq S(x) \wedge F & \text{(ind. hyp.)} \\ \approx_\Sigma & F & \text{(A2)} \end{aligned}$$

□

Proposition 1. If G_d contains a cycle of negative cost, then $d \approx_\Sigma F$.

Proof. If G_d contains a cycle of cost $k < 0$, then \hat{d} contains conjuncts $S^{m_i}(p_i) \preceq S^{n_i}(p_{i+1})$ for $i = 0, \dots, \ell$ with $p_{\ell+1} = p_0$ and $(n_0 - m_0) + \cdots + (n_\ell - m_\ell) = k$. Using (A10,11), we get $\hat{d} \approx_\Sigma \hat{d} \wedge S^{-k}(p_0) \preceq p_0$. So by Lemma 1 and (A2), $\hat{d} \approx_\Sigma F$. Hence, $d \approx_\Sigma \hat{d} \approx_\Sigma F$. □

Assume G_d does not contain cycles of negative cost. Let ∞ denote infinity: $k < \infty$ for all $k \in \mathbb{Z}$. We define a mapping $\mu_d : P \times P \rightarrow \mathbb{Z} \cup \{\infty\}$, where $\mu_d(p, q)$ is the minimum cost of any path from p to q in G_d , or the value ∞ if no path from p to q exists in G_d . Such a minimum exists, owing to the absence of cycles of negative cost. Proposition 2 below expresses that if $p, q \in P$ with $\mu_d(p, q) < \infty$, then d implies $p \preceq S^{\mu_d(p, q)}(q)$ if $\mu_d(p, q) \geq 0$ or $S^{-\mu_d(p, q)}(p) \preceq q$ if $\mu_d(p, q) < 0$. For notational convenience, inequalities $S^m(p) \preceq q$ are written as $p \preceq S^{-m}(q)$. First, we need to prove two lemmas.

Lemma 2. $x \preceq S^n(x) \approx_\Sigma T$ for $n \geq 0$.

Proof. By induction on n . The base case $n = 0$ is (A7). For the inductive case, assume $x \preceq S^n(x) \approx_\Sigma T$ can be derived. Using (A10), $S(x) \preceq S^{n+1}(x) \approx_\Sigma T$. Also, by (A8), $x \preceq S(x) \approx_\Sigma T$. So using (A1,11), $x \preceq S^{n+1}(x) \approx_\Sigma T$. \square

Lemma 3. $x \preceq S^k(y) \wedge x \preceq S^\ell(y) \approx_\Sigma x \preceq S^k(y)$ if $k \leq \ell$.

Proof. We focus on the case where $k \geq 0$ and $\ell \geq 0$. The cases where $k < 0$ or $\ell < 0$ can be treated in a similar fashion.

By Lemma 2 and (A10), $S^k(y) \preceq S^\ell(y) \approx_\Sigma T$. Thus,

$$\begin{aligned} & x \preceq S^k(y) \wedge x \preceq S^\ell(y) \\ \approx_\Sigma & x \preceq S^k(y) \wedge S^k(y) \preceq S^\ell(y) \wedge x \preceq S^\ell(y) & \text{(A1)} \\ \approx_\Sigma & x \preceq S^k(y) \wedge S^k(y) \preceq S^\ell(y) & \text{(A11)} \\ \approx_\Sigma & x \preceq S^k(y) & \text{(A1)} \end{aligned}$$

\square

Proposition 2. If G_d does not contain cycles of negative cost, then

$$d \approx_\Sigma \bigwedge_{\{p,q\} \in P \times P \mid \mu_d(p,q) < \infty} p \preceq S^{\mu_d(p,q)}(q).$$

Proof. Using induction on the number of edges in a path of minimal cost from p to q , we can derive $\hat{d} \approx_\Sigma \hat{d} \wedge p \preceq S^{\mu_d(p,q)}(q)$ for all $p, q \in P$ with $\mu_d(p, q) < \infty$. The base case with zero edges follows by (A7), the base case with one edge by (A10), and the inductive case with more than one edge by (A10,11). So

$$d \approx_\Sigma \hat{d} \approx_\Sigma \hat{d} \wedge \bigwedge_{\{p,q\} \in P \times P \mid \mu_d(p,q) < \infty} p \preceq S^{\mu_d(p,q)}(q).$$

Since $\mu_d(p, q)$ is the minimum cost of any path from p to q , Lemma 3 together with (A10) can be used to eliminate all conjuncts of \hat{d} from the latter term. \square

We are now ready to prove the converse of Proposition 1.

Proposition 3. If G_d does not contain cycles of negative cost, then d is satisfiable.

Proof. Let $\sigma(x) = -\mu_d(0, x)$ for all $x \in V_{bc}$. Note that $\mu_d(0, x) \leq 0$, and so $0 \leq \sigma(x) < \infty$ for all $x \in V_{bc}$. For we took care to include conjuncts $0 \preceq x$ in \hat{d} , which give rise to edges of zero cost from 0 to x in G_d . Note furthermore that $\sigma(0) = 0 = -\mu_d(0, 0)$, owing to the absence of cycles of negative cost in G_d .

For any pair $p, q \in P$ with $\mu_d(p, q) < \infty$, we have $\mu_d(0, q) \leq \mu_d(0, p) + \mu_d(p, q)$; i.e., $\sigma(p) \leq \sigma(q) + \mu_d(p, q)$. So in view of Proposition 2, σ satisfies d . \square

Now let us return to the Boolean terms b and c , not containing any Boolean variables, that coincide under all valuations. Clearly, b and c only contain variables from V_{bc} . Let us assume that b and c are satisfiable. In that case, by Proposition 1, G_b and G_c do not contain cycles of negative cost, so μ_b and μ_c are well-defined.

Proposition 4. If b is satisfiable and b and c coincide under all valuations, then $\mu_b(p, q) = \mu_c(p, q)$ for all $p, q \in P$.

Proof. Assume, toward a contradiction, that $\mu_b(p, q) > \mu_c(p, q)$ for some pair $p, q \in P$. Choose an M sufficiently large (this is specified below). We define for $r \in P$:

$$\nu(r) = \begin{cases} \mu_b(p, r) & \text{if } \mu_b(p, r) < \infty \\ M + \min\{\mu_b(s, r) \mid s \in P\} & \text{otherwise} \end{cases}$$

$\mu_b(0, r) \leq 0$, so $\nu(r) < \infty$ for all $r \in P$. Since $\mu_b(p, q) > \mu_c(p, q)$, by choosing M sufficiently large, we can enforce

$$\nu(q) > \mu_c(p, q).$$

Furthermore, by taking $M \geq \mu_b(p, s) - \mu_b(t, s)$ for all $s, t \in P$ with $\mu_b(p, s) < \infty$ and $\mu_b(t, s) < \infty$, we get

$$\nu(s) \leq M + \mu_b(t, s) \text{ for all } s, t \in P \text{ with } \mu_b(t, s) < \infty.$$

Finally, we have

$$\nu(s) \leq \nu(r) + \mu_b(r, s) \text{ for all } r, s \in P \text{ with } \mu_b(r, s) < \infty.$$

Because if $\mu_b(p, r) < \infty$, then $\mu_b(p, s) < \infty$, so $\nu(s) = \mu_b(p, s) \leq \mu_b(p, r) + \mu_b(r, s) = \nu(r) + \mu_b(r, s)$. Moreover, if $\mu_b(p, r) = \infty$, then $\nu(r) = M + \mu_b(t, r)$ for some $t \in P$, so $\nu(s) \leq M + \mu_b(t, s) \leq M + \mu_b(t, r) + \mu_b(r, s) = \nu(r) + \mu_b(r, s)$.

We define $\sigma(x) = \nu(0) - \nu(x)$ for $x \in V_{bc}$. Since $\mu_b(0, x) \leq 0$, we have $\nu(x) \leq \nu(0) + \mu_b(0, x) \leq \nu(0)$, and so $\sigma(x) \geq 0$ for all $x \in V_{bc}$. Furthermore, $\sigma(0) = 0 = \nu(0) - \nu(0)$. We show that σ satisfies b , but not c .

Consider any pair $r, s \in P$ with $\mu_b(r, s) < \infty$. Since $\nu(s) \leq \nu(r) + \mu_b(r, s)$, we have $\sigma(r) = \nu(0) - \nu(r) \leq \nu(0) - (\nu(s) - \mu_b(r, s)) = \sigma(s) + \mu_b(r, s)$. So σ satisfies $r \preceq S^{\mu_b(r, s)}(s)$. Hence, in view of Proposition 2, σ satisfies b .

$\nu(p) = \mu_b(p, p) = 0$ and $\nu(q) > \mu_c(p, q)$, so $\sigma(p) = \nu(0) = \sigma(q) + \nu(q) > \sigma(q) + \mu_c(p, q)$. In view of Proposition 2, σ does not satisfy c , contradicting that b and c coincide under all valuations. \square

Theorem 1. (A1-11) constitute a basis for \mathcal{A} .

Proof. Terms of sort natural are of the form $S^n(0)$ or $S^n(x)$. So if two terms of sort natural coincide under all valuations, then clearly they are syntactically equal.

Let b_0 and c_0 be Boolean terms that coincide under all valuations. They are of the form $b \wedge u_1 \wedge \cdots \wedge u_m$ and $c \wedge v_1 \wedge \cdots \wedge v_n$, respectively, where b and c do not contain Boolean variables. By setting all u_i and v_j to T , one obtains that b and c coincide under all valuations. We distinguish two cases.

- CASE 1: b and c are not satisfiable.
Then according to Proposition 3, G_b and G_c contain cycles of negative cost. So by Proposition 1, $b \approx_\Sigma F \approx_\Sigma c$. Hence, by (A2), $b_0 \approx_\Sigma F \approx_\Sigma c_0$.
- CASE 2: b and c are satisfiable.
Then clearly $u_1 \wedge \cdots \wedge u_m$ and $v_1 \wedge \cdots \wedge v_n$ must consist of the same Boolean variables. By (A5), $u_1 \wedge \cdots \wedge u_m \approx_\Sigma v_1 \wedge \cdots \wedge v_n$.

By Proposition 1, G_b and G_c do not contain cycles of negative cost. So by Proposition 2 together with Proposition 4,

$$\begin{aligned} b &\approx_{\Sigma} \bigwedge_{\{(p,q) \in P \times P \mid \mu_b(p,q) < \infty\}} p \preceq S^{\mu_b(p,q)}(q) \\ &\approx_{\Sigma} \bigwedge_{\{(p,q) \in P \times P \mid \mu_c(p,q) < \infty\}} p \preceq S^{\mu_c(p,q)}(q) \approx_{\Sigma} c. \end{aligned}$$

So $b_0 = b \wedge u_1 \wedge \cdots \wedge u_m \approx_{\Sigma} c \wedge v_1 \wedge \cdots \wedge v_n \approx_{\Sigma} c_0$. □

Remark. Theorem 1 should be contrasted with the situation that is obtained if \leq is replaced by eq , representing the equality function of type $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{B}$. The resulting algebra has no finite basis. We will not elaborate on the proof, but only notice that it can be constructed along the same lines as the proof of Theorem 4, by giving an infinite basis containing, among others, the true equations $eq(S^n(x), x) = F$ for $n \geq 1$.

3. Addition

We introduce addition $+$: $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, and continue to study the algebra $\mathcal{B} = \langle \mathbb{B}, \mathbb{N}; T, F, 0, 1, \wedge, +, \preceq \rangle$. Note that $S(t)$ is definable as $t + 1$. Let nt denote $t + \cdots + t$ (n summands) for $n \geq 1$.

(A1-11) and the axioms below, consisting of (B1-5) together with the infinite family (B6. n) for $n > 1$, are true in \mathcal{B} . Let Σ' denote the equational base consisting of (A1-11), (B1-5) and (B6. n). We show that Σ' constitutes a basis for \mathcal{B} .

$$\begin{aligned} \text{(B1)} \quad & x + 0 \approx x \\ \text{(B2)} \quad & x + y \approx y + x \\ \text{(B3)} \quad & (x + y) + z \approx x + (y + z) \\ \text{(B4)} \quad & x \preceq y \approx x + z \preceq y + z \\ \text{(B5)} \quad & 2x \preceq 2y + 1 \approx 2x \preceq 2y \\ \text{(B6.}n\text{)} \quad & nx \preceq ny \approx x \preceq y \end{aligned}$$

$s \approx_{\Sigma'}$ denotes that the equation can be derived from Σ' . In view of (B2,3), terms of sort natural can be taken modulo associativity and commutativity of addition. Note that (B6. n) for $n > 1$ can be derived from (B6. p) for primes p . Because if $mx \preceq my \approx_{\Sigma'} x \preceq y$ and $nx \preceq ny \approx_{\Sigma'} x \preceq y$ can be derived, then $(mn)x \preceq (mn)y \approx_{\Sigma'} nx \preceq ny \approx_{\Sigma'} x \preceq y$. Furthermore, (A10) follows directly from (B4).

Finally, (A7,8) can be derived from the other axioms:

$$\begin{aligned} & x \preceq x & x \preceq x + 1 \\ \approx_{\Sigma'} \quad & 0 + x \preceq 0 + x & \approx_{\Sigma'} \quad 0 + x \preceq x + 1 & \text{(B1)} \\ \approx_{\Sigma'} \quad & 0 \preceq 0 & \approx_{\Sigma'} \quad 0 \preceq 1 & \text{(B4)} \\ \approx_{\Sigma'} \quad & T & \approx_{\Sigma'} \quad T & \text{(A6)} \end{aligned}$$

Let b and c be Boolean terms, not containing any Boolean variables, that coincide under all valuations. We develop the machinery to prove $b \approx_{\Sigma'} c$.

Using (A7,9) and (B1), b and c can be written as conjunctions of inequalities

$$n_0 + n_1x_1 + \cdots + n_kx_k \preceq n'_0 + n'_1x_1 + \cdots + n'_kx_k.$$

Furthermore, by (A6), conjuncts $0 \preceq x_i$ for $i = 1, \dots, k$ can be added, so that negative solutions are ruled out. This enables us to use the following results from integer programming.

Chvátal [7] and Schrijver [20] showed that a number of proof principles, called *cutting planes*, suffice to equate each pair of such conjuncts that have the same solutions over the *integer* domain. Here, we follow the presentation of cutting planes in [6].

Definition 1. The three principles underlying *cutting plane proofs* are:

- (1) $nx \preceq ny + r \Rightarrow x \preceq y$ for $r = 0, \dots, n - 1$;
- (2) $x_1 \preceq y_1 \wedge x_2 \preceq y_2 \Rightarrow x_1 + x_2 \preceq y_1 + y_2$;
- (3) $x \preceq m \Rightarrow x \preceq n$ if $m \leq n$.

Theorem 2. (Chvátal '73, Schrijver '80) *Let b and c be conjuncts of inequalities $s \preceq t$, where the s and t are of the form $n_0 + n_1x_1 + \dots + n_kx_k$.*

- (1) *If b is not satisfiable for any valuation over the integer domain, then there is a cutting plane proof for $b \approx F$.*
- (2) *If b and c are satisfiable, and they coincide under all valuations over the integer domain, then there is a cutting plane proof for $b \approx c$.*

We proceed to show that the three principles in Definition 1 can be derived from the axioms. The third principle is a straightforward consequence of Lemma 2 and (A1,10,11). We focus on deriving the first two principles.

Proposition 5. $x_1 \preceq y_1 \wedge x_2 \preceq y_2 \approx_{\Sigma'} x_1 \preceq y_1 \wedge x_2 \preceq y_2 \wedge x_1 + x_2 \preceq y_1 + y_2$.

Proof.

$$\begin{aligned}
& x_1 \preceq y_1 \wedge x_2 \preceq y_2 \\
\approx_{\Sigma'} & x_1 \preceq y_1 \wedge x_2 \preceq y_2 \wedge x_1 + x_2 \preceq y_1 + x_2 \wedge x_2 + y_1 \preceq y_2 + y_1 & \text{(A5, B4)} \\
\approx_{\Sigma'} & x_1 \preceq y_1 \wedge x_2 \preceq y_2 \wedge x_1 + x_2 \preceq y_1 + x_2 \wedge x_2 + y_1 \preceq y_2 + y_1 \\
& \wedge x_1 + x_2 \preceq y_1 + y_2 & \text{(A11)} \\
\approx_{\Sigma'} & x_1 \preceq y_1 \wedge x_2 \preceq y_2 \wedge x_1 + x_2 \preceq y_1 + y_2 & \text{(B4, A5)}
\end{aligned}$$

□

Lemma 4. $2^m x \preceq 2^m y + r \approx_{\Sigma'} x \preceq y$ for $r = 0, \dots, 2^m - 1$.

Proof. By induction on m . The base case $m = 0$ is trivial. For the inductive case, assume the result holds for m , and let $0 \leq r < 2^{m+1}$. Then $0 \leq r/2 < 2^m$, so if r is even,

$$\begin{aligned}
2^{m+1}x \preceq 2^{m+1}y + r & \approx_{\Sigma'} 2^m x \preceq 2^m y + r/2 & \text{(B6.2)} \\
& \approx_{\Sigma'} x \preceq y & \text{(ind. hyp.)}
\end{aligned}$$

and if r is odd,

$$\begin{aligned}
2^{m+1}x \preceq 2^{m+1}y + r & \approx_{\Sigma'} 2^{m+1}x \preceq 2^{m+1}y + r - 1 & \text{(B5)} \\
& \approx_{\Sigma'} 2^m x \preceq 2^m y + (r - 1)/2 & \text{(B6.2)} \\
& \approx_{\Sigma'} x \preceq y & \text{(ind. hyp.)}
\end{aligned}$$

□

Proposition 6. $nx \preceq ny + r \approx_{\Sigma'} x \preceq y$ for $r = 0, \dots, n-1$.

Proof. Fix an m with $2^m \geq n$. Let $2^m r = qn - r'$ with $0 \leq r' < n$. Note that $qn \leq 2^m(n-1) + n - 1 < 2^m n$, so $0 \leq q < 2^m$. Hence,

$$\begin{aligned} nx \preceq ny + r &\approx_{\Sigma'} 2^m nx \preceq 2^m(ny + r) + r' && \text{(Lem. 4)} \\ &\approx_{\Sigma'} 2^m nx \preceq 2^m ny + qn && \\ &\approx_{\Sigma'} 2^m x \preceq 2^m y + q && \text{(B6.n)} \\ &\approx_{\Sigma'} x \preceq y && \text{(Lem. 4)} \end{aligned}$$

□

Theorem 3. (A1-6,9,11), (B1-5) and (B6.p) for primes p constitute a basis for \mathcal{B} .

Proof. Using (B1), each term of sort natural can be written in the form $n_0 + n_1 x_1 + \dots + n_k x_k$. Clearly, if two terms of the latter form coincide under all valuations, then they are syntactically equal (modulo associativity and commutativity of addition).

Let b_0 and c_0 be Boolean terms that coincide under all valuations. They are of the form $b \wedge u_1 \wedge \dots \wedge u_m$ and $c \wedge v_1 \wedge \dots \wedge v_n$, respectively, where b and c do not contain Boolean variables. Clearly, b and c coincide under all valuations. We distinguish two cases.

- CASE 1: b and c are not satisfiable.

Then b and c do not have solutions in the natural numbers. By (A6), conjuncts $0 \preceq x$ can be added to guarantee that they do not have integer solutions. Since the three proof principles of cutting planes can be derived from the axioms, Theorem 2(1) implies $b \approx_{\Sigma'} F \approx_{\Sigma'} c$. Hence, by (A2), $b_0 \approx_{\Sigma'} F \approx_{\Sigma'} c_0$.

- CASE 2: b and c are satisfiable.

Then $u_1 \wedge \dots \wedge u_m$ and $v_1 \wedge \dots \wedge v_n$ consist of the same Boolean variables. By (A5), $u_1 \wedge \dots \wedge u_m \approx_{\Sigma'} v_1 \wedge \dots \wedge v_n$. Furthermore, by (A6), conjuncts $0 \preceq x$ can be added to guarantee that b and c coincide over the integer domain. So, according to Theorem 2(2), $b \approx_{\Sigma'} c$. So $b_0 = b \wedge u_1 \wedge \dots \wedge u_m \approx_{\Sigma'} c \wedge v_1 \wedge \dots \wedge v_n \approx_{\Sigma'} c_0$.

□

We proceed with a proof that \mathcal{B} does not have a finite basis, by exhibiting algebras M_p for primes p such that:

- (1) for each finite set of equations that are true in \mathcal{B} there is a prime p such that the equations hold in M_p ;
- (2) (B6.p) does not hold in M_p for primes p .

The algebras M_p consist of the Booleans \mathbb{B} together with \mathbb{N}^3 . The two-sorted algebra of Booleans and naturals is embedded in M_p : conjunction is interpreted as usual, 0 and 1 are mapped to $(0, 0, 0)$ and $(1, 0, 0)$, respectively, and $+$ and \preceq are defined on \mathbb{N}^3 as follows.

$$\begin{aligned} (m_0, m_1, m_2) + (n_0, n_1, n_2) &= (m_0 + n_0, m_1 + n_1, m_2 + n_2) \\ (m_0, m_1, m_2) \preceq (n_0, n_1, n_2) &= m_0 \leq n_0 \wedge (m_1 + n_1 < m_2 + n_2 \\ &\quad \vee (m_1 + n_1 = m_2 + n_2 \wedge m_1 \equiv n_1 \pmod{p})) \end{aligned}$$

Note that if p divides n , then (B6. n) does not hold in M_p . Namely, $(0, n, 0) \preceq (0, 0, n)$, because $n \equiv 0 \pmod{p}$, but not $(0, 1, 0) \preceq (0, 0, 1)$.

Proposition 7. *Let p be a prime. (A1-6,9,11), (B1-5) and (B6. n) for $n \not\equiv 0 \pmod{p}$ hold in M_p .*

Proof. (A1-5) and (B1-3) clearly hold in M_p . We focus on the remaining axioms.

- (A6): $(0, 0, 0) \preceq (n_0, n_1, n_2) = 0 \leq n_0 \wedge (0 < n_1 + n_2 \vee (0 = n_1 + n_2 \wedge 0 \equiv n_1 \pmod{p})) = T$. Because if $0 = n_1 + n_2$, then $n_1 = 0$.
- (A9): $(n_0, n_1, n_2) + (1, 0, 0) \preceq (n_0, n_1, n_2) = F$, since $n_0 + 1 > n_0$.
- (A11): If $(\ell_0, \ell_1, \ell_2) \preceq (m_0, m_1, m_2) \wedge (m_0, m_1, m_2) \preceq (n_0, n_1, n_2) = T$, then $\ell_0 \leq m_0 \leq n_0$, and either $\ell_1 + \ell_2 < m_1 + m_2 \leq n_1 + n_2$, $\ell_1 + \ell_2 \leq m_1 + m_2 < n_1 + n_2$, or $\ell_1 + \ell_2 = m_1 + m_2 = n_1 + n_2$ with $\ell_1 \equiv m_1 \equiv n_1 \pmod{p}$. This implies $\ell_0 \leq n_0$, and either $\ell_1 + \ell_2 < n_1 + n_2$, or $\ell_1 + \ell_2 = n_1 + n_2$ with $\ell_1 \equiv n_1 \pmod{p}$. So $(\ell_0, \ell_1, \ell_2) \preceq (n_0, n_1, n_2) = T$.
- (B4): $(\ell_0, \ell_1, \ell_2) + (n_0, n_1, n_2) \preceq (m_0, m_1, m_2) + (n_0, n_1, n_2) = \ell_0 + n_0 \leq m_0 + n_0 \wedge (\ell_1 + n_1 + \ell_2 + n_2 < m_1 + n_1 + m_2 + n_2 \vee (\ell_1 + n_1 + \ell_2 + n_2 = m_1 + n_1 + m_2 + n_2 \wedge \ell_1 + n_1 \equiv m_1 + n_1 \pmod{p})) = \ell_0 \leq m_0 \wedge (\ell_1 + \ell_2 < m_1 + m_2 \vee (\ell_1 + \ell_2 = m_1 + m_2 \wedge \ell_1 \equiv m_1 \pmod{p})) = (\ell_0, \ell_1, \ell_2) \preceq (m_0, m_1, m_2)$.
- (B5): $(2m_0, 2m_1, 2m_2) \preceq (2n_0 + 1, 2n_1, 2n_2) = 2m_0 \leq 2n_0 + 1 \wedge (2m_1 + 2m_2 < 2n_1 + 2n_2 \vee (2m_1 + 2m_2 = 2n_1 + 2n_2 \wedge 2m_1 \equiv 2n_1 \pmod{p})) = 2m_0 \leq 2n_0 \wedge (2m_1 + 2m_2 < 2n_1 + 2n_2 \vee (2m_1 + 2m_2 = 2n_1 + 2n_2 \wedge 2m_1 \equiv 2n_1 \pmod{p})) = (2m_0, 2m_1, 2m_2) \preceq (2n_0, 2n_1, 2n_2)$.
- (B6. n) with $n \not\equiv 0 \pmod{p}$: $(n\ell_0, n\ell_1, n\ell_2) \preceq (nm_0, nm_1, nm_2) = n\ell_0 \leq nm_0 \wedge (n\ell_1 + n\ell_2 < nm_1 + nm_2 \vee (n\ell_1 + n\ell_2 = nm_1 + nm_2 \wedge n\ell_1 \equiv nm_1 \pmod{p})) = \ell_0 \leq m_0 \wedge (\ell_1 + \ell_2 < m_1 + m_2 \vee (\ell_1 + \ell_2 = m_1 + m_2 \wedge \ell_1 \equiv m_1 \pmod{p})) = (\ell_0, \ell_1, \ell_2) \preceq (m_0, m_1, m_2)$.

□

Theorem 4. *\mathcal{B} does not have a finite basis.*

Proof. Let E be a finite set of equations that are true in \mathcal{B} . By Theorem 3, the equations in E can be derived from (A1-6,9,11), (B1-5) and (B6. n) for $n \in \mathbb{N}$. Since each proof of an equation in E requires only a finite number of applications of axioms, the equations in E can be derived from (A1-6,9,11), (B1-5) and (B6. n) where n ranges over a finite set N of naturals. Fix a prime p that does not divide any of the elements of N . By Proposition 7, all equations in E hold in M_p . Since (B6. p) does not hold in M_p , it cannot be derived from E . □

4. Negation

We expand \mathcal{B} with negation $\neg : \mathbb{B} \rightarrow \mathbb{B}$, to get the algebra $\mathcal{C} = \langle \mathbb{B}, \mathbb{N}; T, 0, 1, \neg, \wedge, +, \preceq \rangle$. Note that F is definable as $\neg T$. We use \vee for notational convenience, where $b \vee c$ represents $\neg(\neg b \wedge \neg c)$. As binding convention, \neg binds stronger than \wedge . The axioms (C1-4) below are true in \mathcal{C} .

Let Γ denote the equational base consisting of (A1-11), (B1-4) and (C1-4). We show that Γ constitutes a basis for \mathcal{C} .

$$\begin{aligned} \text{(C1)} \quad & u \wedge \neg u \approx F \\ \text{(C2)} \quad & \neg\neg u \approx u \\ \text{(C3)} \quad & u \wedge (v \vee w) \approx (u \wedge v) \vee (u \wedge w) \\ \text{(C4)} \quad & \neg(x \preceq y) \approx y + 1 \preceq x \end{aligned}$$

$s \approx_{\Gamma} t$ denotes that the equation can be derived from (A1-11), (B1-4) and (C1-4). Clearly, $x \vee T \approx_{\Gamma} T$ and $x \vee F \approx_{\Gamma} x$. (A9) can be derived from the other axioms (note that the derivation of (A7) in the previous section did not use (A9)):

$$\begin{aligned} x + 1 \preceq x &\approx_{\Gamma} \neg\neg(x + 1 \preceq x) && \text{(C2)} \\ &\approx_{\Gamma} \neg(x + 1 \preceq x + 1) && \text{(C4)} \\ &\approx_{\Gamma} F && \text{(A7)} \end{aligned}$$

As in Section 3, we want to show that the three proof principles of cutting planes can be derived from the axioms. In Section 3 it was shown that these three principles can be derived from (A1-11), (B1-5) and (B6. n) for $n \geq 1$. So it suffices to derive (B5) and (B6. n) for $n \geq 1$ from our current set of axioms.

Lemma 5. $x \preceq y \approx_{\Gamma} x \preceq y \wedge nx \preceq ny$ for $n \geq 1$.

Proof. By induction on n . The base case $n = 1$ follows by (A5). For the inductive case, note that the derivation of Proposition 5 used neither (B5) nor (B6. n), so that it also follows from our current set of axioms.

$$\begin{aligned} x \preceq y &\approx_{\Gamma} x \preceq y \wedge nx \preceq ny && \text{(ind. hyp.)} \\ &\approx_{\Gamma} x \preceq y \wedge nx \preceq ny \wedge x + nx \preceq y + ny && \text{(Prop. 5)} \\ &\approx_{\Gamma} x \preceq y \wedge (n + 1)x \preceq (n + 1)y && \text{(ind. hyp.)} \end{aligned}$$

□

Note that in order to prove $b \approx_{\Gamma} c$, it suffices to derive $b \wedge \neg c \approx_{\Gamma} F$ and $\neg b \wedge c \approx_{\Gamma} F$, because then we have $b \approx_{\Gamma} b \wedge \neg(b \wedge \neg c) \approx_{\Gamma} b \wedge (\neg b \vee c) \approx_{\Gamma} b \wedge c \approx_{\Gamma} (b \vee \neg c) \wedge c \approx_{\Gamma} \neg(\neg b \wedge c) \wedge c \approx_{\Gamma} c$.

Proposition 8. $nx \preceq ny \approx_{\Gamma} x \preceq y$ for $n \geq 1$.

Proof.

$$\begin{aligned} &nx \preceq ny \wedge \neg(x \preceq y) \\ \approx_{\Gamma} &nx \preceq ny \wedge y + 1 \preceq x && \text{(C4)} \\ \approx_{\Gamma} &nx \preceq ny \wedge y + 1 \preceq x \wedge ny + n \preceq nx && \text{(Lem. 5)} \\ \approx_{\Gamma} &nx \preceq ny \wedge y + 1 \preceq x \wedge ny + n \preceq nx \wedge ny + n \preceq ny && \text{(A11)} \\ \approx_{\Gamma} &F && \text{(Lem. 1, A2)} \end{aligned}$$

$$\begin{aligned} &\neg(nx \preceq ny) \wedge x \preceq y \\ \approx_{\Gamma} &\neg(nx \preceq ny) \wedge x \preceq y \wedge nx \preceq ny && \text{(Lem. 5)} \\ \approx_{\Gamma} &F && \text{(C1, A2)} \end{aligned}$$

□

Proposition 9. $2x \preceq 2y + 1 \approx_{\Gamma} 2x \preceq 2y$.

Proof. By Proposition 8 it suffices to derive $2x \preceq 2y + 1 \approx_{\Gamma} x \preceq y$.

$$\begin{aligned}
& 2x \preceq 2y + 1 \wedge \neg(x \preceq y) \\
\approx_{\Gamma} & 2x \preceq 2y + 1 \wedge y + 1 \preceq x && \text{(C4)} \\
\approx_{\Gamma} & 2x + 1 \preceq 2y + 2 \wedge 2y + 2 \preceq 2x && \text{(A10, Prop. 8)} \\
\approx_{\Gamma} & 2x + 1 \preceq 2y + 2 \wedge 2y + 2 \preceq 2x \wedge 2x + 1 \preceq 2x && \text{(A11)} \\
\approx_{\Gamma} & F && \text{(A2, 9)}
\end{aligned}$$

$$\begin{aligned}
& \neg(2x \preceq 2y + 1) \wedge x \preceq y \\
\approx_{\Gamma} & 2y + 2 \preceq 2x \wedge x \preceq y && \text{(C4)} \\
\approx_{\Gamma} & y + 1 \preceq x \wedge x \preceq y && \text{(Prop. 8)} \\
\approx_{\Gamma} & y + 1 \preceq x \wedge x \preceq y \wedge y + 1 \preceq y && \text{(A11)} \\
\approx_{\Gamma} & F && \text{(A2, 9)}
\end{aligned}$$

□

Theorem 5. (A1-6,11), (B1-4) and (C1-4) constitute a basis for \mathcal{C} .

Proof. Using (B1), each term of sort natural can be reduced to the form $n_0 + n_1x_1 + \dots + n_kx_k$. If two terms of the latter form coincide under all valuations, then they are syntactically equal. Let b and c be Boolean terms that coincide under all valuations; we prove $b \approx_{\Gamma} c$. As stated earlier, it suffices to derive $b \wedge \neg c \approx_{\Gamma} F$ and $\neg b \wedge c \approx_{\Gamma} F$.

$b \wedge \neg c$ and $\neg b \wedge c$ can be written in disjunctive normal form using (A7,9), (B1) and (C2-4), where each disjunct is of the form

$$s_1 \preceq t_1 \wedge \dots \wedge s_n \preceq t_n \wedge u_1 \wedge \dots \wedge u_m \wedge \neg v_1 \wedge \dots \wedge \neg v_\ell$$

with the s_i and t_j of the form $n_0 + n_1x_1 + \dots + n_kx_k$. It suffices to equate each of the disjuncts above to F .

- (1) If $u_i = v_j$ for some i and j , then the disjunct above can be equated to F using (C1) and (A2).
- (2) If the u_i and v_j are all distinct, then clearly $s_1 \preceq t_1 \wedge \dots \wedge s_n \preceq t_n$ must be unsatisfiable, so it does not have solutions in the natural numbers. By (A6), conjuncts $0 \preceq x_i$ can be added for $i = 1, \dots, k$, so we can even assume that there are no integer solutions. Since the three principles of cutting planes can be derived from the axioms, it follows that by Theorem 2(1) that $s_1 \preceq t_1 \wedge \dots \wedge s_n \preceq t_n \approx_{\Gamma} F$. So the disjunct above can be equated to F with the help of (A2).

□

5. If-then-else

The algebra \mathcal{D} consists of \mathcal{C} extended with the conditional operator $\triangleleft \triangleright : \mathbb{N} \times \mathbb{B} \times \mathbb{N} \rightarrow \mathbb{N}$, where $s \triangleleft b \triangleright t$ evaluates to s or t if b is true or false, respectively. Bloom and Tindell [5] and Manes [15] provided finite bases for algebras with an if-then-else

construct, but without \preceq . Note that $\max\{x, y\}$ is captured by $x \triangleleft y \preceq x \triangleright y$; so \mathcal{D} encompasses the max-plus algebra of [1], which was proved not to have a finite basis.

The axioms (D1-4) below are true in the algebra $\mathcal{D} = \langle \mathbb{B}, \mathbb{N}; T, 0, 1, \neg, \wedge, +, \preceq, \triangleleft, \triangleright \rangle$. Let Γ' be the equational base consisting of (A1-11), (B1-4), (C1-4) and (D1-4). We show that Γ' constitutes a basis for \mathcal{D} .

$$\begin{aligned} \text{(D1)} \quad & x \triangleleft T \triangleright y \approx x \\ \text{(D2)} \quad & x \triangleleft (y \preceq x) \triangleright y \approx y \triangleleft (x \preceq y) \triangleright x \\ \text{(D3)} \quad & (x \triangleleft u \triangleright y) + z \approx (x + z) \triangleleft u \triangleright (y + z) \\ \text{(D4)} \quad & (x \triangleleft u \triangleright y) \preceq z \approx (u \wedge x \preceq z) \vee (\neg u \wedge y \preceq z) \end{aligned}$$

$s \approx_{\Gamma'} t$ denotes that the equation can be derived from (A1-11), (B1-4), (C1-4) and (D1-4).

As a first step, we will show that $\triangleleft \triangleright$ can be eliminated from Boolean terms. To this end we will use the following equations from left to right as rewrite rules:

$$(x \triangleleft u \triangleright y) + z \approx (x + z) \triangleleft u \triangleright (y + z) \quad (1)$$

$$z + (x \triangleleft u \triangleright y) \approx (z + x) \triangleleft u \triangleright (z + y) \quad (2)$$

$$(x \triangleleft u \triangleright y) \preceq z \approx (u \wedge x \preceq z) \vee (\neg u \wedge y \preceq z) \quad (3)$$

$$z \preceq (x \triangleleft u \triangleright y) \approx (u \wedge z \preceq x) \vee (\neg u \wedge z \preceq y) \quad (4)$$

Lemma 6. *Equations (1)–(4) can be derived from Γ' .*

Proof. (1) and (3) correspond to (D3) and (D4), respectively. (2) can be derived by (D3) and (B2). (4) is derived as follows:

$$\begin{aligned} & z \preceq (x \triangleleft u \triangleright y) \\ \approx_{\Gamma'} & \neg \neg (z \preceq (x \triangleleft u \triangleright y)) \quad (\text{C2}) \\ \approx_{\Gamma'} & \neg (x \triangleleft u \triangleright y) + 1 \preceq z \quad (\text{C4}) \\ \approx_{\Gamma'} & \neg ((x + 1 \triangleleft u \triangleright y + 1) \preceq z) \quad (\text{D3}) \\ \approx_{\Gamma'} & \neg ((u \wedge x + 1 \preceq z) \vee (\neg u \wedge y + 1 \preceq z)) \quad (\text{D4}) \\ \approx_{\Gamma'} & (u \wedge z \preceq x) \vee (\neg u \wedge z \preceq y) \quad (\text{Thm. 5}) \end{aligned}$$

Note that the last step involves a true equation without if-then-else. \square

Next we show that the rewrite system above doesn't admit infinite rewrite sequences.

Lemma 7. *The rewrite rules (1)–(4) are terminating.*

Proof. This is a standard application of the lexicographic path order in term rewriting [?]. The required precedence of function symbols is $+ \succ \triangleleft \triangleright$ and $\leq \succ \{\vee, \text{and}, \neg\}$. \square

Lemma 8. *For every Boolean term s of \mathcal{D} there exists a term t without $\triangleleft \triangleright$, such that $s \approx_{\Gamma'} t$.*

Proof. By the previous lemmas, we find a term $s \approx_{\Gamma'} r$, to which no rewrite rule (1)–(4) is applicable. Assume towards a contradiction that r contains $\triangleleft \triangleright$. Let r' be the largest subterm of r that starts with $\triangleleft \triangleright$. This is a natural, so it has a function

symbol above it. The only candidates are $\triangleleft \triangleright$ itself — which contradicts maximality of r' — or $+$ or \leq (which contradicts that (1)–(4) are not applicable). \square

Theorem 6. Γ' constitutes a basis for \mathcal{D} .

Proof. Let b and c be Boolean terms that coincide under all valuations. By the previous lemma we find b' and c' without $\triangleleft \triangleright$, with $b \approx_{\Gamma'} b'$ and $c \approx_{\Gamma'} c'$. Then b' and c' coincide under all valuations, hence by Theorem 5, $b' \approx_{\text{Gamma}} c'$. This proves $b \approx_{\Gamma'} c$.

Next, let s and t be terms of sort natural that coincide under all valuations. Then both $s \preceq t \approx T$ and $t \preceq s \approx T$ are true Boolean equations in \mathcal{D} , so by the previous paragraph, $s \preceq t \approx_{\Gamma'} T$ and $t \preceq s \approx_{\Gamma'} T$. We derive $s \approx_{\Gamma'} t$ as follows:

$$s \approx_{\Gamma'} s \triangleleft T \triangleright t \approx_{\Gamma'} s \triangleleft (t \preceq s) \triangleright t \approx_{\Gamma'} t \triangleleft (s \preceq t) \triangleright s \approx_{\Gamma'} t \triangleleft T \triangleright s \approx_{\Gamma'} t.$$

\square

6. Conclusion

In this paper we studied the finite base problem for a number of algebras. In particular, we studied two-sorted algebras of natural numbers, extended with a Boolean algebra. It appears that adding a Boolean algebra to a given algebra causes some algebras to become finitely based. In particular, the algebra \mathcal{B} with operations $+$, \leq , \wedge was not finitely based, but adding \neg (thus obtaining full Boolean algebra) makes it finitely based. Similarly, the algebra based on \max , $+$ is not finitely based, but adding Boolean algebra made it finitely based.

It is well-known that the fact whether an algebra is finitely based depends in a subtle way on the precise set of operators. The following relationship with recursion theory is shown in [4]. If the set of true equations in some algebra is recursively enumerable, then there exists an ω -complete theory (i.e., finite basis), provided hidden sorts and function symbols are allowed. Two remarks are in order. The specification is complete with respect to equations in the smaller language; the hidden function symbols are auxiliary operators only. Second, from the general construction it is not clear how to obtain a concrete finite axiomatization.

The algebras that we deal with are clearly recursively enumerable (even recursive). So, by the recursion theoretical argument it is immediately clear that by extending the signature, we can get a finitely based algebra. Our contribution is that for certain examples, we give concrete axiomatizations in a systematic way, by adding a Boolean algebra. Moreover, we prove completeness of the algebra over the full new language.

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