Monochromatic Van der Waerden Triples

By Van der Waerden's theorem, any 2-colouring of the integers admits arbitrarily long arithmetic progressions of the same colour. In particular, there are monochromatic arithmetic progressions of length three in any 2-colouring of $\{1, 2, \ldots, n\}$ for $n \geq 9$. Motivated by a question of Graham, we obtain a lower bound on the asymptotic number of monochromatic three-term arithmetic progressions that are inevitable in any 2-colouring of $\{1, 2, \ldots, n\}$. The analogous question on Schur triples $\{x, y, x+y\}$ was completely solved by Robertson and Zeilberger [2], and independently by Schoen [3], who established that the minimum number of Schur triples is $N^2/22 + O(N)$.

The best known bounds on this problem are due to Parrilo, Robertson and Saracino [1] who showed, with the aid of a Maple program and a G5 Macintosh server, that the minimum number of monochromatic three-term arithmetic progressions is between $N^2/(19.5629...)$ and $N^2/(18.7350...)$. Note that the upper bound implies that random colourings, which yield $N^2/16$ monochromatic progressions, are not optimal. Our technique is inspired by Roth's lower bound on the discrepancy of arithmetic progressions. While no records have been harmed in the production of this bound, we hope that our approach is of some interest.

Let χ be the given 2-colouring, whose domain is extended to $\mathbb{Z}$ by defining it to be 0 outside $[1, N]$. Let $\gamma_d(m) = 1$ if $m = -d, 0$ or d, and 0 otherwise. Define

$$F_d(m) = (\chi \star \gamma_d)(m) = \chi(m-d) + \chi(m) + \chi(m+d)$$

By Parseval's identity, we have

$$\sum_{m=1}^{n} |F_d(m)|^2 = \int_0^1 |\hat{F}_d(t)|^2 \, dt = \int_0^1 |\hat{\gamma}_d(t)|^2 |\hat{\chi}(t)|^2 \, dt$$

Note that $\hat{\gamma_d}(t) = 1 + 2\cos(2\pi dt)$. Thus,

$$|\hat{\gamma}_d(t)|^2 = 3 + 2\cos(4\pi dt) + 4\cos(2\pi dt)$$

Since the Dirichlet kernel given by

$$D_M(\theta) = \sum_{d=1}^{M} \cos(d\theta)$$

can be negative, we work with the non-negative Fejér kernel, given by

$$F_M(\theta) = \sum_{d=1}^{M} \left(1 - \frac{d}{M}\right) \cos(d\theta)$$

Accordingly, for suitably chosen c and $M = cN$, we get

$$\sum_{d=1}^{cN} \left(1 - \frac{d}{cN}\right) \sum_{m=1}^{n} |F_d(m)|^2 = \int_0^1 \left(1 - \frac{d}{cN}\right) |\hat{\gamma}_d(t)|^2 |\hat{\chi}(t)|^2 \, dt \geq \frac{3}{2} cN^2$$

The upshot is that the average value of $|F_d(m)|^2$ is at least $3/2$. Since $|F_d(m)|^2$ equals 9 if the three-term progression is monochromatic and 1 if it is not, we expect this to yield a lower bound on monochromatic arithmetic progressions. However, the fact that some arithmetic progressions go out of bounds leads to complications, and calls for a judicious choice of c.

Let G_d be the number of 3-term monochromatic progressions with common difference d. Accounting for the contributions of $2d$ "fake" three-term arithmetic progressions, we have,

$$\sum_{d=1}^{cN} \left(1 - \frac{d}{cN}\right) (9G_d + (N - 2d - G_d) + 8d) \geq \frac{3}{2} cN^2$$

Rearranging terms, we get,

$$\frac{3}{2} cN^2 - \sum_{d=1}^{cN} (N + 6d) \left(1 - \frac{d}{cN}\right) \leq \sum_{d=1}^{cN} \left(1 - \frac{d}{cN}\right) 8G_d$$

A straightforward computation shows that

$$\frac{1}{cN} \sum_{d=1}^{cN} \left(1 - \frac{d}{cN}\right) \left(\frac{G_d}{N}\right) \geq \frac{1-c}{8}$$

Let $x = d/(cN) \in [0, 1]$ and let $f(x)$ denote the distribution function of G_d. Since $G_d \leq N - 2d$ for all d, we also require that $f(x) \leq 1 - 2cx$ for all

x. This leads to the following question:

Given $\int_0^1 (1-x)f(x)\,dx \geq (1-c)/8$ and $f(x) \leq 1-2cx, 0 \leq x \leq 1$, how small can $\int_0^1 f(x)\,dx$ be?

It is easy to see that "shifting the entire weight to the left" results in the least possible integral. Thus if a satisfies

$$\int_0^a (1-x)(1-2cx)\,dx = (1-c)/8$$

then we have

$$\int_0^1 f(x)\,dx \geq \int_0^a (1-2cx)\,dx = a(1-ac)$$

Thus we are interested in the solution of the following problem:

Maximise $ac(1-ac)$ subject to

$$(2c/3)a^3 - (2c+1)(a^2/2) + a - (1-c)/8 = 0$$

We use Lagrange multipliers to solve this problem. Define

$$L(a,c) = ac - a^2c^2 + \lambda[(2c/3)a^3 - ((2c+1)/2)a^2 + a - (1-c)/8]$$

Setting the partial derivatives equal to zero, we get

$$\begin{aligned} \frac{\partial L}{\partial a} &= (c - \lambda a + \lambda)(1-2ac) = 0 \\ \frac{\partial L}{\partial c} &= (a - 2a^2c) + \lambda[(2/3)a^3 - a^2 + (1/8)] = 0 \end{aligned}$$

Simplification leads to the following equations:

$$\begin{aligned} 32a^4 - 32a^3 + 60a^2 - 48a + 3 &= 0 \\ 24a(1-a) + c(32a^3 - 24a^2 - 3) &= 0 \end{aligned}$$

Solving these, we get $a = 0.0681...$ and $c = 0.4911...$, yielding $N^2/(30.9335...)$ monochromatic three-term arithmetic progressions.

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References

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