Fractions with Bounded Partial Quotients

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Zaremba [5] conjectured that for every positive integer n, there exists $m \leq n,\ (m,n) = 1$ such that all partial quotients in the continued fraction expansion of $\frac{m}{n}$ are bounded by some absolute constant B, possibly 5.

For $c \geq 2$, let $S_c(x)$ denote the set of integers $m \leq x$, such that there is a reduced fraction $\frac{a}{m} = [0; a_1, a_2, \ldots, a_n]$, with $a_i \leq c$ for all i. Zaremba's conjecture states that $|S_5(x)| = x$ for all positive integers x.

Sander [4] showed that $|S_c(x)| \geq \frac{1}{\sqrt{2c}} x^{\alpha_c}$ where $\alpha_c = \frac{1}{2} - \frac{1}{c^2}$

Theorem 1 $|S_c(x)| >> x^{\beta_c}$, where $\beta_2 = 0.508..., \beta_3 = 0.644...,$ $\beta_4 = 0.692..., \beta_5 = 0.712..., \beta_6 = 0.719...$ and $\beta_k = 0.722...$ for $k \geq 7$.

Proof We present the details for the case $c = 2$. Let $U = \{u_n\}_{n=0}^{\infty}$ be a sequence defined as follows:

$$u_0 = u_1 = 1; u_{2k} = u_k + u_k^*; u_{2k+1} = 2u_k + u_k^*, \text{ where } u_k^* \doteq u_{\lfloor \frac{k}{2} \rfloor}$$

The sequence $\{u_n\}_{n=1}^{\infty}$ can be thought of as a binary tree, rooted at $u_1 = 1$. Note that $S_2(x) = \{n | n \leq x, n = u_k \text{ for some } k \geq 1\}$.

Let $\mu_r \doteq \frac{r+\sqrt{r^2+4}}{2}$, and $\beta_2 = \frac{\ln 2}{\ln \mu_1 + \ln \mu_2} = 0.508...$, so that $(\mu_1 \mu_2)^{2\beta_2} = 4$. Further, let $A_2 = (\frac{\sqrt{5}\mu_1\mu_2}{3})^{2\beta_2} = 2.966...$ and $y = A_2 x^{2\beta_2}$

Given an integer t, define $f_2(t) = (p,q)$ if the binary expansion of t has exactly p zeroes and q ones. Let $Q_2(\ell) = \{t : f_2(t) = (p,q)$ with $p \geq q,$ and $p + q = 2\ell - 1\}$.

Let $M = \lfloor \log_4 y \rfloor$, so that $4^M \leq y < 4^{M+1}$. Now $|Q_2(M)| > 2^{2M-3} \geq \frac{y}{8}$.

We claim that for all $k \in Q_2(M), u_k \leq x$.

For $1 \leq j \leq 2$, let $M_j = \begin{bmatrix} 0 & 1 \\ 1 & j \end{bmatrix}$

For $k \geq 2$, let $\mathbf{v_k}$ denote $[u_k^* \ u_k]^T$. Note that $M_0 \mathbf{v_k} = \mathbf{v_{2k}}$ and $M_1 \mathbf{v_k} = \mathbf{v_{2k+1}}$.

Let $s \in Q_2(M)$. Then $\mathbf{v_s} = \prod_{i=1}^{2M-2} M_{\epsilon_i} [1 \ 1]^T, \epsilon_i \in \{0, 1\}$.

Let $P_0 = \{i : \epsilon_i = 0\}$ and $P_1 = \{i : \epsilon_i = 1\}$. Clearly, $|P_0| + |P_1| = 2M - 2$. Since $s \in Q_2(M)$, we have $|P_0| \leq |P_1|$.

Since M_1 and M_2 are symmetric, $||M_1||_2 = \rho(M_1) = \mu_1$ and $||M_2||_2 = \rho(M_2) = \mu_2$. It follows that

$$||\mathbf{v_s}||_2 \leq || \prod_{i=1}^{2M-2} M_{\epsilon_i} ||_2 \ \ ||[1 \ 1]^T||_2 \leq \sqrt{2} \prod_{i=1}^{2M-2} ||M_{\epsilon_i}||_2 \leq \sqrt{2} (\mu_1 \mu_2)^{M-1}$$

Note that $u_s \leq 3u_s^*$. Thus $\frac{\sqrt{10}}{3} u_s \leq ||\mathbf{v_s}||_2 \leq \sqrt{2}(\mu_1 \mu_2)^{M-1}$

Therefore, $u_s \leq \frac{3}{\sqrt{5}\mu_1\mu_2} (\mu_1\mu_2)^M = \frac{3}{\sqrt{5}\mu_1\mu_2} 4^{\frac{M}{2\beta_2}} < \frac{3}{\sqrt{5}\mu_1\mu_2} y^{1/2\beta_2} = x$. This proves our claim.

Thus there are at least $N \doteq \frac{A_2 x^{2\beta_2}}{8}$ elements in U, counting multiplicity, which are less than or equal to x.

Observe that $\mathbf{v_k} = \mathbf{v_\ell}$ if and only if $k = \ell$. If there are fewer than $\sqrt{N}$ elements in $S_2(x)$, then $\exists \, t > \sqrt{N}$, such that $u_{i_1} = u_{i_2} = \ldots = u_{i_t}$. But then $u_{i_1}^*, u_{i_2}^*, \ldots u_{i_t}^*$ are t distinct elements in $S_2(x)$, contradicting our assumption. It follows that $|S_2(x)| >> x^{0.508\ldots}$

The argument is identical for higher values of c, where the sequence resembles a c-ary tree, and we consider numbers whose c-ary expansion has at least as many zeroes as ones, at least as many ones as twos and so on. The value of β_c is maximum for $c = 7$.

While this is an improvement over the results obtained in [4], better estimates for the number of fractions with denominator at most x and partial quotients bounded by c do exist. Cusick [2] showed that this number is asymptotically $x^{2H(c)}$, where $H(c)$ is the Hausdorff dimension of the set of reals in $[0, 1]$ with partial quotients at most c. Hensley [3] showed that $H(2) = 0.531...$, $H(3) = 0.705...$, $H(4) = 0.788...$, $H(5) = 0.836...$, $H(6) = 0.867...$ and in general, $1 - \frac{5}{2c} < H(c) < 1 - \frac{1}{10(c+1)}$. It follows from the proof of Theorem 1 that $\beta_c \geq H(c)$, so that these estimates are superior for all values of c. Even with the easy lower bound $\beta_c \geq 2H(c) - 1$, these estimates give better results for $c \geq 6$.

Let $\overline{S}_c(x)$ denote the set of reduced fractions with denominator at most x and partial quotients bounded on the average by c. Cooper [1] showed that $|\overline{S}_2(x)| >> x^{1.57}$.

Theorem 2 $|\overline{S}_2(x)| >> x^{1.7}$

Proof Define $\alpha_i = 2^{-i}$, so that $\sum_{i=1}^{20} \alpha_i < 1$ and $\sum_{i=1}^{20} i\alpha_i < 2$. Let $\delta = 2^{-42}$. For $k \geq 1$ and $1 \leq j \leq 20$, define $I_j^k = [(\alpha_j - \delta)k, \alpha_j k]$. Note that for sufficiently large k, each of the intervals $I_1^k, \ldots, I_{20}^k$ contain an integer. Choose integers $a_j \in I_j^k$.

Let $s_t \doteq \sum_{j=1}^{t} a_j$. Note that $s_t \in [k(1 - 2^{-t} - t\delta), k(1 - 2^{-t})]$. In particular, $s_{20} > 0.9999k$. Let $\alpha'_j \doteq \frac{a_j}{s_{20}}$. Note that $\alpha'_j < 1.0001\alpha_j$. Let $s_t^* \doteq \sum_{k=t}^{20} a_k > k(2^{1-t} - 2^{-20} - 20\delta)$.

Let $H(x) \doteq -(x \log_2 x + (1 - x) \log_2(1 - x))$ denote the binary entropy function. It is well known, and easy to show, that $\binom{n}{\theta n} \sim 2^{nH(\theta)}$. Thus there are

$$M \doteq \binom{s_1^*}{a_1}\binom{s_2^*}{a_2}\cdots\binom{s_{20}^*}{a_{20}} \sim 2^{\sum_{j=1}^{20} s_j^* H(\frac{a_j}{s_j^*})}$$

sequences $< 0; b_1, b_2, \ldots b_{s_{20}} >$ which contain exactly a_j occurrences of the partial quotient j.

Note that

$$\frac{a_j}{s_j^*} > \frac{(\alpha_j - \delta)k}{s_j^*} > \frac{(\alpha_1 - 2^{j-1}\delta)k}{s_{20}} > \frac{1}{2}$$

But we also have,

$$\frac{a_j}{s_j^*} < \frac{\alpha_j k}{s_j^*} < \frac{\alpha_j k}{\alpha_j(2-2^{j-20})k} = \frac{1}{2-2^{j-20}}$$

Thus,

$$\sum_{j=1}^{20} s_j^* H(\frac{a_j}{s_j^*}) > k\sum_{j=1}^{20}(2^{1-j} - 2^{-20} - 20\delta)H(\frac{1}{2-2^{j-20}}) > 1.995k$$

From what we have seen above all these sequences correspond to distinct rational numbers with denominators at most $x = \nu^{s_{20}} < \nu^k$ where

$$\nu \doteq \prod_{j=1}^{20} \mu_j^{\alpha'_j} \leq \prod_{j=1}^{20} \mu_j^{1.0001\alpha_j} \leq 2.255$$

Thus $M > 3.986^k$ and $x < 2.255^k$. It follows that for sufficiently large x, there are $M > x^{1.7}$ fractions with denominator at most x and average value of partial quotients at most 2.

Note: There is a slight catch. We need all partial quotients to be bounded in average, so the average of the first j partial quotients must be less than 2 for all $j, 1 \leq j \leq 20$. We sort of assumed we need this only for $j = 20$. Since this is like replacing the n^{th} Catalan number by $\binom{2n}{n}$ (see [1], for instance), and we are interested only in asymptotics, we don't really care. All right, I don't.

References

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