MATHEMATICS 300 — FALL 2017

Introduction to Mathematical Reasoning

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HOMEWORK ASSIGNMENT NO. 12, DUE ON WEDNESDAY, DECEMBER 6

This assignment consists of two parts: Part I involves several problems from the notes, and Part II has two problems.

Part I

Problems 52, 56, 60, 63, 64 from the tenth set of lecture notes. (NOTE: I strongly recommend that you practice by doing ***all*** the problems in Part X of the notes. But you are only asked to hand in the five problems I have listed.)

Part II

The problems in this part have to do with functions. So I will first review the basic definitions.

Definition of "function". A function is a set f such that

(i) f is a set of ordered pairs[1],

(ii) for all u, v, w, if $(u, v) \in f$ and $(u, w) \in f$ then $v = w$. $\square$

Definition of "domain" of a function. If f is a function then the domain of f is the set $\mathrm{Dom}(f)$ consisting of all u such that $(u, v) \in f$ for some v.

That is,

$$\mathrm{Dom}(f) = \{u : (\exists v)(u, v) \in f\} .$$

Definition of "value" of a function at a member of its domain. If f is a function and u is a member of $\mathrm{Dom}(f)$, the the value of f at u is the uniuqe v such that $(u, v) \in f$. We write $f(u)$ for the value of f at u. $\square$

[1] That is, f is a set and every member of f is an ordered pair.

NOTE: If f is a function, we think of the pairs (u, v) that belong to f as "input-output pairs": v is "the output that f produces for the input u", or "the value of f at u", and we write $f(u)$ for v. $\square$

NOTE: Why can we talk about v as ***the*** value of f at u? *Because, for every u, v is unique.* Indeed, the condition in the definition of "function" ("if $(u, v) \in f$ and $(u, w) \in f$ then $v = w$") says precisely that, for every u, if there is a v such that $(u, v) \in f$, then there is only one such v.) It follows from this that an ordered pair (u, v) belongs to f if and only if u belongs to $\text{Dom}(f)$ and $v = f(u)$. $\square$

Definition of "function from a set to a set". If f is a function and A, B are sets, we say that <u>f is a function from A to B</u>, or that <u>f maps A to B</u>, and write

$$f : A \to B\,,$$

if

1. $\text{Dom}(f) = A$,
2. $(\forall u \in A) f(u) \in B$.

Definition of "one-to-one function". A function $f : A \to B$ is <u>one-to-one</u> if $(\forall u, v \in A)(u \neq v \Longrightarrow f(u) \neq f(v))$. $\square$

Definition of "function onto a set". A function $f : A \to B$ is <u>onto B</u> if $(\forall b \in B)(\exists a \in A) f(a) = b$. $\square$

Definition of "bijection". A <u>bijection</u> from a set A to a set B is a function $f : A \to B$ which is one-to-one and onto B. $\square$

Problem 1. ***Prove*** that

(#) *$\emptyset : \emptyset \to \emptyset$ and*[2] *$\emptyset$ is a bijection from $\emptyset$ to $\emptyset$.*

This means that you have to prove several things:

1. That $\emptyset$ is a function[3].

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[2] That is, you have to prove that the empty set is a function and that it is a bijection from the empty set to the empty set.

[3] In view of the definition of "function", to prove that $\emptyset$ is a function you have to prove two things: *first* you have to prove that $\emptyset$ is a set of ordered pairs (that is, that every member of $\emptyset$ is an ordered pair), and *then* you have to prove that $(\forall u, v)\Big(\big((u, v) \in f \wedge (u, v) \in f\big) \Longrightarrow v = w\Big)$.

2. That the domain of $\emptyset$ is $\emptyset$.
3. That $\emptyset : \emptyset \to \emptyset$.
4. That $\emptyset$ is one-to-one.
5. That $\emptyset$ is onto B.

Problem 2. ***Prove***, *without using any of the results from the notes based on lists*, the following theorem, which is one of the most basic[4] results of finite set theory:

(#) *If a set A and two nonnegative integers m, n are such that there exists a bijection $f : \mathbb{N}_m \to A$ and there exists a bijection $g : \mathbb{N}_n \to A$, then $m = n$.*

(This was proved in the notes using lists: We proved that *if* $\mathbf{a} = (a_j)_{j=1}^m$ *and* $\mathbf{b} = (b_j)_{j=1}^n$ *are two lists without repetitions of all the members of a set* A*, then* $m = n$, and used this result to justify the definition[5] of $\mathrm{card}(A)$ as ***the*** number n such that there exists a list of length n without repetitions of all the members of A. (In order to be able to talk about *the* number n such that there exists a list of length n without repetitions of all the members of A we have to prove that n is unique, that is, that if n is a number such that there

[4] When I say "basic", I mean "important and used all the time". This does not mean that the theorem is very difficult to prove. (The proof is actually very simple.) A theorem can be both very easy to prove and very important. For example, the fact that $2+2=4$ is very important, and we use it all the time, but the proof is very simple. Cantor's theorem that if X is any set then there does not exist a bijection from X to $\mathcal{P}(X)$ is very simple (the proof is just a few lines) but extremely important. The fact that every natural number is ≥ 1 is very easy to prove, but it is very important: we use it all the time.

[5] We cannnot talk about "*the* number n such that XXXX" unless we know that there exists a *unique* number n such that XXX. For example, we cannot talk about "*the* integer n that divides a number a" because in general there are lots of different integers that divide a number a. But we can talk about *an* integer that divides a. And we cannot talk about *the* city that has more tnan $1,000,000$ people, because there are lots of cities that have more tnan $1,000,000$ people. But we can talk about *the* greatest common divisor of two integers, because we proved that the GCD, when it exists, is unique. And we can talk about *the* capital of New Jersey, because New Jersey has only one capital. Similarly, if we define the meaning of "A has n members"', or "A has cardinality n", as "there is a bijection from $\mathbb{N}_n$ to A", then in order to be able to talk about *the* cardinality of A, we have to prove that n is unique, that is, that if there is a bijection from $\mathbb{N}_n$ to A and there is a bijection from $\mathbb{N}_m$ to A then $m = n$.

exists a list of length n without repetitions of all the members of A, and m is a number such that there exists a list of length m without repetitions of all the members of A, then it follows that $m = n$.)

You are not allowed to use the results from the notes based on lists. But you ***are*** allowed to look at the notes, read the proof given there, and then translate it into function language.

If you do not want to translate the proof given in the notes, here are some hints for proving the result directly:

1. first prove that[6]

   (*) *if X, Y are two sets, $f : X \to Y$ is a bijection, x is a member of X, and y is a member of Y, then you can create another bijection $g : X \to Y$ that sends x to y.*

   (To prove this, first construct, for any two members u, v of Y, a bijection $s_{u,v} : Y \to Y$ that 'swaps" u and v, that is, sends u to v and v to u. Just define $s_{u,v}$ by letting $s_{u,v}(z) = z$ if z is any member of Y other than u or v, $s_{u,v}(u) = v$, and $s_{u,v}(v) = u$. You have to prove that $s_{u,v}$ is a bijection. Then define g by letting $g = s_{f(x),y} \circ f$. You have to prove that g has the properties we want, that is, g is a bijection from X to Y and $g(x) = y$.)

2. Next, prove that

   (**) *if $f : \mathbb{N}_n \to A$ and $g : \mathbb{N}_m \to A$ are bijections, then you can construct a bijection h from $\mathbb{N}_n$ to $\mathbb{N}_m$.*

   (Hint: let $h = g^{-1} \circ f$.)

3. Next, prove that

[6] Here is the idea: suppose you have a set W of women and a set M of men, and the women are dancing with men, each woman is dancing with a man, and the "dancing partner" function f (i.e., the function that sends each woman w to the man $f(w)$ that she is dancing with) is a bijection from W to M. (That is, different women dance with different men, and every man is dancing with some woman.) Suppose a man m and a woman w would like to be dancing with each other. Then you can change the bijection f and create instead a bijection g that corresponds to m and w dancing with each other. The way you do that is by taking the man a with whom w is dancing, and the woman b with whom w is dancing, and swapping partners, that is, having w dance with m and b dance woith a. That's all there is to this step. ***If you think this is difficult stuff, it's because you have not yet realized how simple it is!***

(***) *if h is a bijection from $\mathbb{N}_n$ to $\mathbb{N}_m$, then you can construct a bijection k from $\mathbb{N}_n$ to $\mathbb{N}_m$ such that $k(n) = m$.*

4. Next, prove that

(***) *if k is a bijection from $\mathbb{N}_n$ to $\mathbb{N}_m$ such that $k(n) = m$, then you can construct a bijection ℓ from $\mathbb{N}_{n-1}$ to $\mathbb{N}_{m-1}$.*

5. Now that you have the previous results, prove by induction on n that if there is a bijection from $\mathbb{N}_n$ to $\mathbb{N}_m$ then $n = m$. (Be careful about how you choose the predicate $P(n)$ for your induction! Remember that it has to be a predicate with n as open variable and no other open variables. So $P(n)$ cannot be "if there is a bijection from $\mathbb{N}_n$ to $\mathbb{N}_m$ then $n = m$", and it cannot be "$(\forall n \in \mathbb{N})(\forall m \in \mathbb{N})$ if there is a bijection from $\mathbb{N}_n$ to $\mathbb{N}_m$ then $n = m$".)