MATHEMATICS 300 — FALL 2017
Introduction to Mathematical Reasoning
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HOMEWORK ASSIGNMENT NO. 13, DUE ON WEDNESDAY, DECEMBER 13

This assignment consist of three problems.

In these problems, if a and b are real numbers, we write "$[a, b]$" to denote the closed interval $\{x \in \mathbb{R} : a \leq x \leq b\}$, and "$]a, b[$" to denote the open interval $\{x \in \mathbb{R} : a < x < b\}$.

If A is a subset of $\mathbb{R}$, an interior point of A is a point $a \in A$ such that the open interval $]a - \varepsilon, a + \varepsilon[$ is a subset of A for some positive[1] real number ε.

If A is a set, then the identity map of A is the function I_A such that (1) $\mathrm{Dom}(I_A) = A$, and (2) $I_A(x) = x$ for every $x \in A$.

Problem 1. ***Prove*** that if X is a set then there does not exist a one-to-one function[2] $f : \mathcal{P}(X) \longrightarrow X$.

Problem 2. ***Prove*** that if $f : A \longrightarrow B$, then f is a bijection from A to B if and only if the following is true:

(#) There exists a function $g : B \longrightarrow A$ such that $g \circ f = I_A$ and $f \circ g = I_B$.

Problem 3.

1. Let f, g be the functions defined by

   (i) $\mathrm{Dom}(f) = \mathbb{R}$,

   (ii) $f(x) = \frac{x}{\sqrt{1+x^2}}$ for $x \in \mathbb{R}$,

   (iii) $\mathrm{Dom}(g) =]-1, 1[$,

   iv) $g(y) = \frac{y}{\sqrt{1-y^2}}$ for $y \in]-1, 1[$,

   (a) ***Prove*** that $f : \mathbb{R} \longrightarrow]-1, 1[$, $g :]-1, 1[\longrightarrow \mathbb{R}$, $g \circ f = I_{\mathbb{R}}$, and $f \circ g = I_{]-1,1[}$.

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[1] "Positive" means "> 0".

[2] "$\mathcal{P}(X)$" stands for "the power set of X". By definition, $\mathcal{P}(X)$ is the set of all subsets of X. That is, $\mathcal{P}(X) = \{U : U \subseteq X\}$.

(b) **Conclude** from this that $\mathbb{R}$ and the interval $]-1, 1[$ have the same cardinality.

2. If $a, b \in \mathbb{R}$ and $a < b$, let $f_{a,b}$ be the function with domain $]-1, 1[$, given by $f_{a,b}(x) = a + \frac{1}{2}(b - a)(x + 1)$ for $x \in]-1, 1[$. **Prove** that f is a bijection from $]-1, 1[$ to $]a, b[$, and **conclude** from this that $\mathbb{R}$ and the interval $]a, b[$ have the same cardinality.

3. **Prove** that if S is a subset of $\mathbb{R}$ such that S has an interior point then S has the same cardinality as $\mathbb{R}$. (HINT: Use the Cantor-Schroeder-Bernstein theorem. Construct a one-to-one map[3] from $\mathbb{R}$ to S and a one-to-one map from S to $\mathbb{R}$.)

4. Let $\mathbb{I}$ be the set of all irrational numbers. **Prove** that $\mathbb{I}$ has the same cardinality as $\mathbb{R}$.

   HINT: Think of a very large hotel in which the rooms correspond to the irrational numbers, in the sense that the hotel has a room x for every $x \in \mathbb{I}$. And think of $\mathbb{R}$ as a set of guests: for each real number r, there is a guest, guest r. You want to put each guest in a room. You can start by putting each guest with an irrational number in a room, by putting guest no. x in room no. x if $x \in \mathbb{I}$. And now you have to find rooms for the guests corresponding to the rational numbers, i.e., the members of $\mathbb{Q}$. Since the set $\mathbb{Q}$ is countably infinite, there is a bijection $b : \mathbb{N} \longrightarrow \mathbb{Q}$. So the rational numbers are the numbers $b(1)$, $b(2)$, $b(3)$, and so on. Find a sequence $x_1, x_2, x_3, \ldots$ of pairwise distinct irrational numbers (for example, you could take $x_n = n\sqrt{2}$), and move the guests occupying rooms $x_1, x_2, x_3, \ldots$ to rooms $x_2, x_4, x_6, \ldots$, thereby leaving room for the guests numbered $b(1), b(2), b(3), \ldots$ to be put in rooms $x_1, x_3, x_5, \ldots$.

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[3] "Map" means the same thing as "function".