MATHEMATICS 300 — FALL 2017

Introduction to Mathematical Reasoning

H. J. Sussmann

HOMEWORK ASSIGNMENT NO. 7, DUE ON WEDNESDAY, OCTOBER 25

In these problems "$\mathbb{N} \cup \{0\}$" stands for the set of all nonegative integers, so that $\mathbb{N} \cup \{0\}$ is the union of $\mathbb{N}$, the set of all natural numbers, and the set $\{0\}$ whose only member is the number 0. Then

$$\mathbb{N} \cup \{0\} = \{n \in \mathbb{Z} : n \geq 0\}\,.$$

Recall that, by definition,

$$\sum_{k=1}^{0} a_k = 0\,, \qquad \prod_{k=1}^{0} a_k = 1\,, \qquad 0! = 1\,, \qquad a^0 = 1\,.$$

1. ***Prove*** that if n is a natural number then

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}\,.$$

2. ***Prove*** that if $n \in \mathbb{N} \cup \{0\}$ then

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}\,.$$

3. ***Prove*** that if r is real number, and $n \in \mathbb{N} \cup \{0\}$, then

$$\sum_{k=0}^{n} r^k = \begin{cases} \frac{1-r^{n+1}}{1-r} & \text{if} \quad r \neq 1 \\ n+1 & \text{if} \quad r = 1\,. \end{cases}$$

(NOTE: "$\sum_{k=0}^{n} a_k$" is defined inductively, exactly as "$\sum_{k=1}^{n} a_k$" was. The only difference is that we start at 0 rather than 1. So the definition is:

$$\begin{aligned}\sum_{k=0}^{0} a_k &= a_0\,,\\ \sum_{k=0}^{n+1} a_k &= \Big(\sum_{k=0}^{n} a_k\Big) + a_{n+1} \qquad \text{for } n \in \mathbb{N} \cup \{0\}\,.\end{aligned}$$

4. ***Prove*** that if $n \in \mathbb{N} \cup \{0\}$ then

$$\prod_{j=1}^{n} \Big(1 - \frac{1}{j+1}\Big) = \frac{1}{n+1}\,.$$

5. ***Prove*** that if $n \in \mathbb{N} \cup \{0\}$ then

$$\prod_{\ell=1}^{n} (2\ell - 1) = \frac{(2n)!}{2^n n!}\,.$$

6. ***Prove*** that

$$\sum_{j=1}^{n} \frac{1}{\sqrt{j}} > \sqrt{n} \qquad \text{for all } n \in \mathbb{N} \text{ such that } n \geq 2\,.$$

7. Book, problem 11 on page 126.

A hint for Problem 7

Let $P(n)$ be the sentence "every tournament with n players has a top player". You want to prove $(\forall n \in \mathbb{N}) P(n)$ by induction.

For the ***basis step***, remember that

A sentence of the form

$$(\forall x)(x \in S \Longrightarrow A(x)) \tag{0.1}$$

is true if the set S ***is the empty set.***

Sentences of the form (0.1) are said to be ***vacuously true***, that is, true because the set S is "vacuous", i.e., empty.

We discussed in class the reason that (0.1) is true: to prove that the statement "$(\forall x)(x \in S \Longrightarrow A(x))$" is true, we have to prove that the statement "$x \in S \Longrightarrow A(x)$" is true for every x. Let x be an arbitrary thing. Then "$x \in S$" is false, because S has no members, so x is not a member of S. Since "$x \in S$" is false, the implication "$x \in S \Longrightarrow A(x)$" is true.

For the ***inductive step***, you want to take an arbitrary natural number n, and prove the implication "$P(n) \Longrightarrow P(n+1)$". For that purpose, you assume that $P(n)$ is true, and try to prove that $P(n+1)$ is true. So we find ourselves in the following situation: we know that

(*) every tournament with n players has a top player,

and we want to prove that

(**) every tournament with $n+1$ players has a top player,

In order to prove (**), we let T be an arbitrary tournament with $n+1$ players, and we must prove that T has a top player. Since we can use (*), the natural thing to do is this:

- From T, which is a tournament with $n+1$ players, construct a tournament S with n players.
- Using (*), conclude that S has a top player.
- Then use the top player of S to get a top player of T, by either:
  - proving that the top player of S is a top player of T

  or

  - constructing, from the top player of S, a top player of T.

In order to construct an n-players tournament S from the $n+1$-players tournament T, the most natural thing to do is to pick one player of T and remove that player.

So the proposed strategy for the proof of $P(n+1)$, assuming $P(n)$, would be as follows:

(1) Pick one player[1] from the set of players of T, call this player p, and remove p from the set of players of T, thus obtaining a tournament S with n players.

(2) Using (*), conclude that S has a top player.

(3) Pick a top player[2] of S and call it q.

(4) Then use the fact that q is a top player of S to prove that T has a top player. And several things may happen:

  (I) Maybe we can prove that q itself must be a top player of T.

  (II) Maybe we can use the fact that q is a top player of S to prove that some other player of T—for example p—is a top player of T.

  (III) Maybe we can prove that either q is a top player of T or some other player of T—for example p—is a top player of T.

What you have to do is this

1. First, you have to figure out how to choose the player p of T that you are going to remove. It may be that

   a. You can just pick p any way you want, and then from the fact that S has a top player you will be able to prove that T has a top player.

   Or, maybe,

   b. You cannot just pick p in any way you want, but you have to be smart and make an intelligent choice of p.

---

[1]Notice that here we are applying Rule $\exists_{use}$, the rule for using existential sentences: if you know that $(\exists x)A(x)$, then you can introduce an object, call this object a, and stipulate that $A(a)$. In our case, $A(x)$ is the sentence "x is a player of T". Since T has $n+1$ players, it follows that T has at least one player, so the sentence "$(\exists x)A(x)$" is true. Then we are picking an object, calling it p, and stipulating that $A(p)$, i.e., that p is a player of T.

[2]Notice that here we are applying again Rule $\exists_{use}$, the rule for using existential sentences: if you know that $(\exists x \in C)A(x)$, then you can introduce an object, call this object a, and stipulate that $a \in C$ and $A(a)$. In our case, C is the set of players of S, and $A(x)$ is the sentence "x is a top player of S". (*) tells us that S has a top player, i.e., that $(\exists x \in C)A(x)$. Then we are picking an object, calling it q, and stipulating that $q \in C$ (i.e., q is a player of S) and $A(q)$, i.e., q is a top player of S.

2. Second, once you have decided how to choose p, and used it to construct S and find a top player of S called q, you have to figure out how to prove that T has a top player.

3. And it may happen that these two things are related. For example, it could happen that if you choose p to be just an arbitrary player of T, then you cannot prove that T has a top player, but if you choose p in a smart way, then maybe you will be able to prove that T has a top player. And you may even be able to prove that q is a top player of T.

A suggestion: Start by picking a player p of T You can do this by writing

Let p be a player of T that satisfies the following condition:

And leave some blank space after that (about four or five lines), so that later, once you know what condition on p you need, you will be able to go back and fill in the blank, and end up with "Let p be a player of T that satisfies the following condition: XXX." (And instead of "XXX" you will write the condition that p has to satisfy, once you know what that condition is.

Then remove p from T, thus constructing the new tournament S.

Then pick a top player of S (which you can do thanks to the inductive hypothesis) and call it q.

Then try to prove that q is a top player of T. You will not be able to, but you will see that your proof that q is a top player of T does work, provided that p satisfies some extra condition K.

Then go back to the first step, fill in the blank by choosing p in such a way that p satisfies condition K.

And make sure that you prove that there does exist a player that satisfies the condition. (This is important: if you cannot prove that there exists a player of T that satisfies Condition K, then you cannot apply Rule $\exists_{use}$ and pick a player of T that satisfies conditon K.)

Then you can easily finish your proof.

WARNING: Here is an example of the kind of thing that could go wrong. You will obviously discover that the following condition K_{bad} works:

(K_{bad}) p is beaten by all the other players of T.

This condition works perfectly. (Proof: Once you know that q is a top player of S, it follows that for every player s of S such that $s \neq q$, either q beats s

or q beats some player that beats s. So the only thing missing to prove that q is a top player of T is to show that q beats p or beats some player that beats p. But we know that p is beaten by all the players of T other than p. So in particular q beats p, and this proves that q is a top player of T, and we are dobe.)

The trouble with this argument is this: *there is no reason to believe that a player that loses to all the other players of T exists.* So we cannot prove that there exists a player of T that satisfies condition K_{bad}. And then we are not allowed to apply Rule $\exists_{use}$ and pick a player that satisfies condition K_{bad} and call it p.

CONCLUSION: The condition K that you need cannot be a simple, naïve condition such as K_{bad}. You need something more sophisticated. And for that you have to **THINK**.