MATHEMATICS 300 — FALL 2017

Introduction to Mathematical Reasoning

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HOMEWORK ASSIGNMENT NO. 8, DUE ON WEDNESDAY, NOVEMBER 1

These problems deal with sets of real numbers, i.e., subsets of \mathbb{R} . (Here \mathbb{R} is the set of all real numbers.) In particular, every subset of \mathbb{N} or of \mathbb{Z} or of¹ \mathbb{Q} is a subset of \mathbb{R} , so some of the sets we will be dealing with will be subsets of \mathbb{N} of \mathbb{Z} or \mathbb{Q} .

So the sets we will be dealing with are members of $\mathcal{P}(\mathbb{R})$, the power set of \mathbb{R} , that is, the set of all subsets of \mathbb{R} .

For any set U, the power set $\mathcal{P}(U)$ is equipped with four important binary operations² namely,

- **union** (i.e., the operation that produces, for each A, B in $\mathcal{P}(U)$, the set $A \cup B$).
- *intersection* (i.e., the operation that produces, for each A, B in $\mathcal{P}(U)$, the set $A \cap B$).
- difference (i.e., the operation that produces, for each A, B in $\mathcal{P}(U)$, the set A B).
- symmetric difference (i.e., the operation that produces, for each A, B in $\mathcal{P}(U)$, the set $A\Delta B$),

and one unary operation³, namely,

• complementation (i.e., the operation that produces, for each A in $\mathcal{P}(U)$, the set A^c , called the complement of A relative to U.

¹Remember that \mathbb{Q} is the set fo all rational numbers. So if x is an arbitrary object then $x \in \mathbb{Q}$ if and only if $x \in \mathbb{R}$ and $(\exists m \in \mathbb{N})(\exists n \in \mathbb{N})(n \neq 0 \land x = \frac{m}{n})$.

²That is, operations with two arguments.

³That is an operation with one argument.

These operations make sense for members of any power set $\mathcal{P}(U)$, for any set U. In addition, when U is \mathbb{R} , then he special fact that \mathbb{R} itself has the operations of addition and multiplication allows us to define two operations on the set $\mathcal{P}(\mathbb{R})$ of subsets of \mathbb{R} .

These operations are:

• set addition, defined as follows: if A, B are subsets of \mathbb{R} , then the sum of A and B is the set A + B defined as follows:

$$A + B = \{x \in \mathbb{R} : (\exists a \in A) (\exists b \in B) x = a + b.$$

In other words: A + B is the set of all sums a + b, $a \in A$, $b \in B$.

• set multiplication, defined as follows: if A, B are subsets of \mathbb{R} , then the product of A and B is the set $A \cdot B$ defined as follows:

$$A \cdot B = \{ x \in \mathbb{R} : (\exists a \in A) (\exists b \in B) x = a \cdot b .$$

In other words: $A \cdot B$ is the set of all products $a \cdot b$, $a \in A$, $b \in B$.

Problem 1. A subset S of \mathbb{R} is <u>bounded</u> if there exists a real number C such that

$$(\forall x)(x \in S \Longrightarrow |x| \le C)$$
.

Prove that

- 1. The empty set is bounded.
- 2. If $A \subseteq \mathbb{R}$, $B \subseteq A$, and A is bounded, then B is bounded.
- 3. If A and B are subsets of \mathbb{R} and A and B are bounded, then the following six sets are bounded:
 - (a) $A \cup B$,
 - (b) $A \cap B$,
 - (c) A B,
 - (d) $A\Delta B$,
 - (e) A + B,
 - (f) $A \cdot B$.

but

• The complement A^c is not bounded.

HINT: You should use the following two facts about the absolute value function:

1. If x, y are real numbers, then

$$|xy| = |x| \cdot |y|.$$

2. The *triangle inequality*: If x, y are real numbers, then

$$|x+y| \le |x|+|y|.$$

Problem 2. A subset S of \mathbb{R} is <u>bounded above</u> if there exists a real number C such that

$$(\forall x)(x \in S \Longrightarrow x \le C)$$

Prove or disprove each of the following statements:

- 1. If a subset A is bounded, then A is bounded above.
- 2. If a subset A is bounded above, then A is bounded.
- 3. The empty set is bounded above.
- 4. If $A \subseteq \mathbb{R}$, $B \subseteq A$, and A is bounded above, then B is bounded above.
- 5. If $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$, and A and B are bounded above, then $A \cup B$ is bounded above.
- 6. If $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$, and A and B are bounded above, then $A \cap B$ is bounded above.
- 7. If $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$, and A and B are bounded above, then A B is bounded above.
- 8. If $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$, and A and B are bounded above, then $A\Delta B$ is bounded above.
- 9. If $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$, and A and B are bounded above, then A + B is bounded above.

- 10. If $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$, and A and B are bounded above, then $A \cdot B$ is bounded above.
- 11. If $A \subseteq \mathbb{R}$ and A is bounded above, then the complement A^c is not bounded above.

Problem 3.

- 1. **Prove** Theorem I below. (To understand this, you have to read the explanation of "partitions" below.)
- 2. (In this list of questions, if A is a set then " $\mathcal{P}(A)$ " stands for "the power set of A", that is, the set $\{X : X \subseteq A\}$.) **Indicate** which of the following are true and which are false (if $b \in \mathbb{N}$):
 - (a) $\mathbf{P}_b \in \mathbb{Z}$,
 - (b) $\mathbf{P}_b \subseteq \mathbb{Z}$,
 - (c) $\mathbf{P}_b \in \mathcal{P}(\mathbb{Z}),$
 - (d) $\mathbf{P}_b \subseteq \mathcal{P}(\mathbb{Z}),$
 - (e) $\mathbf{P}_b \in \mathcal{P}(\mathcal{P}(\mathbb{Z})),$
 - (f) $\mathbf{P}_b \subseteq \mathcal{P}\Big(\mathcal{P}(\mathbb{Z})\Big),$
 - (g) \mathbf{P}_b has b members,
 - (h) $\mathbf{P}_b = b.$
- 3. If $b_1 \in \mathbb{N}$, $b_2 \in \mathbb{N}$, then the condition that b_1 divides b_2 (i.e., that $b_1|b_2$) is equivalent to a condition C_{b_1,b_2} about the partitions \mathbf{P}_{b_1} . \mathbf{P}_{b_2} . (This means "for all $b_1, b_2 \in \mathbb{N}$, $b_1|b_2 \iff C_{b_1,b_2}$ ".) *Indicate* which of the following is the condition C_{b_1,b_2} , and *sketch the proof* that the condition you chose is equivalent to " $b_1|b_2$ ".
 - (a) $\mathbf{P}_{b_1} \subseteq \mathbf{P}_{b_2}$,
 - (b) $\mathbf{P}_{b_2} \subseteq \mathbf{P}_{b_1}$,
 - (c) $(\forall X)(X \in \mathbf{P}_{b_1} \Longrightarrow X \in \mathbf{P}_{b_2}).$
 - (d) $(\forall X)(X \in \mathbf{P}_{b_2} \Longrightarrow X \in \mathbf{P}_{b_1}).$
 - (e) $(\forall X) \Big(X \in \mathbf{P}_{b_1} \Longrightarrow (\exists Y \in \mathbf{P}_{b_2}) X \subseteq Y \Big),$

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(f)
$$(\forall X) \Big(X \in \mathbf{P}_{b_2} \Longrightarrow (\exists Y \in \mathbf{P}_{b_1}) X \subseteq Y \Big).$$

(HINT: Think of the example $b_1 = 3, b_2 = 6$.)

Partitions

If you are asked to define "partition", the first two questions that you have to ask yourself is this: *is "partition" a predicate or a term?*, and *how many arguments does it have, and what kinds of things are they?* The answer is: "partition" is a *2-argument predicate*: we say things like "**P** is a partition of *A*". And both arguments are *sets*, that is, we talk about a set **P** being a partition of *A*. And, furthermore, the first argument **P** must be a set of subsets of the second argument *A*, i.e., a set whose members are subsets of *A*.

And now that we know what kind of thing "partition" is, we can write the definition:

Definition. A partition of a set A is a set \mathbf{P} such that:

- 1. Every member of \mathbf{P} is a nonempty subset of A.
- 2. If X, Y are members of **P** and $X \neq Y$ then $X \cap Y = \emptyset$.
- 3. If a is an arbitrary member of A, then $a \in X$ for some member X of **P**.

Example 1. Let A be the set of all people who live in the United States. Let us pretend, to make this example easier, that all the people who live in the U.S. live in one of the 50 states. (That is, we pretend that here are no territories such as Puerto Rico or Guam, which are part of the U.S. but are not in one of the states.) For each state s, let X_s be the set of all people who live in s. (For example: $X_{\text{New Jersey}}$ is the set of all the people who live in New Jersey; X_{Alabama} is the set of all the people who live in Alabama; and so on.) Let **P** be the set whose members are the 50 sets X_s . That is, if we let S be the set whose members are the 50 states:

$$\mathbf{P} = \left\{ x : (\exists s \in S) x = X_s \right\}.$$

Then **P** is a partition of A. (You should make sure that you understand how to prove this, although I am not asking you to hand in the proof.)

Example 2. Let \mathcal{E} be the set of all even integers, and let \mathcal{O} be the set of all odd integers. That is,

$$\mathcal{E} = \{ n \in \mathbb{Z} : 2|n\}, \\ \mathcal{O} = \{ n \in \mathbb{Z} : 2|n-1\}.$$

Let

$$\mathbf{P} = \{\mathcal{E}, \mathcal{O}\}\,,\,$$

so **P** is the two-member set whose members are the sets \mathcal{E} and \mathcal{O} .

Then **P** is a partition of \mathbb{Z} .

Proof: We have to prove that

- 1. Every member of \mathbf{P} is a nonempty subset of \mathbb{Z} .
- 2. If X, Y are members of **P** and $X \neq Y$ then $X \cap Y = \emptyset$.
- 3. If n is an arbitrary member of \mathbb{Z} , then $n \in X$ for some member X of **P**.

Condition 1 is easy: There are two members of \mathbf{P} , namely, \mathcal{E} and \mathcal{O} , and they are both nonempty: \mathcal{E} is nonempty because, for example, $2 \in \mathcal{E}$, and \mathcal{O} is nonempty because, for example, $1 \in \mathcal{O}$.

To prove Condition 2 we have to show that $\mathcal{E} \cap \mathcal{O} = \emptyset$. But that follows from the fact that no integer can be both even and odd, which tells us precisely that no integer n can belong to both \mathcal{E} and \mathcal{O} , that is, that no integer n can belong to $\mathcal{E} \cap \mathcal{O}$.

To prove Condition 3 we have to show that $\mathcal{E} \cup \mathcal{O} = \mathbb{Z}$. But that follows from the fact that every integer is even or odd, which tells us precisely that every integer *n* belongs to \mathcal{E} or to \mathcal{O} , that is, every integer belongs to $\mathcal{E} \cup \mathcal{O}$, so $\mathcal{E} \cup \mathcal{O} = \mathbb{Z}$. **Example 3.** For each natural number b, and each integer r such that $0 \le r < b$, define a subset $E_{b,r}$ of \mathbb{Z} by letting

$$E_{b,r} = \{n \in \mathbb{Z} : b|n-r\}.$$

In other words, $E_{b,r}$ is the set of all integers n such that the remainder of dividing n by b is r. So, for example,

- $E_{3,0}$ is the set of all integers that are divisible by 3. So the members of $E_{3,0}$ are 0, 3, -3, 6, -6, 9, -9 and so on.
- $E_{3,1}$ is the set of all integers n such that the remainder of dividing n by 3 is 1. So the members of $E_{3,1}$ are 1, 4, -2, 7, -5, 10, -8 and so on.
- $E_{3,2}$ is the set of all integers n such that the remainder of dividing n by 3 is 2. So the members of $E_{3,2}$ are 2, 5, -1, 8, -4, 11, -7 and so on.

For each natural number b, we define a set \mathbf{P}_b of subsets of \mathbb{Z} as follows: \mathbf{P}_b is the set whose members are the sets $E_{b,0}, E_{b,1}, \ldots, E_{b,b-1}$. In other words:

$$\mathbf{P}_b = \{ X : (\exists r \in \mathbb{Z}) (0 \le r < b \land X = E_{b,r} \}.$$

Theorem I. For every natural number b, the set \mathbf{P}_b is a partition of \mathbb{Z} . *Proof.* **YOU DO IT.**