MATHEMATICS 300 — FALL 2018

Introduction to Mathematical Reasoning

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HOMEWORK ASSIGNMENT NO. 4, DUE ON TUESDAY, OCTOBER 9

Problem 1. Book, Page 81, problems 5, 6, 9(b), 10.

Problem 2.

1. ***Prove*** that if n is an integer then there exist unique integers q, r such that $n^2 = 4q + r$ and $r = 0 \vee r = 1$. (Hint: use the division theorem, i.e., the theorem that the book calls the "division algorithm".)

2. ***Prove*** that if m, n are integers then there exist unique integers q, r such that $m^2 + n^2 = 4q + r$ and $r = 0 \vee r = 1 \vee r = 2$.

3. ***Prove*** that if m, n are integers then there exist unique integers q, r such that $m^2 - n^2 = 4q + r$ and $r = 0 \vee r = 1 \vee r = 3$.

4. ***Prove*** that if a is an integer such that $a = 4k + 3$ for some integer k then a cannot be written as the sum of two squares. (In formal language: $(\forall a \in \mathbb{Z})\Big((\exists k \in \mathbb{Z})a = 4k + 3 \Longrightarrow \sim (\exists m \in \mathbb{Z})(\exists n \in \mathbb{Z})m^2 + n^2 = a\Big)$.)

5. ***Prove*** that if a is an integer such that $a = 4k+2$ for some integer k (that is, a is even but not divisible by 4) then a cannot be written as the difference of two squares. (In formal language: $(\forall a \in \mathbb{Z})\Big((\exists k \in \mathbb{Z})a = 4k + 2 \Longrightarrow \sim (\exists m \in \mathbb{Z})(\exists n \in \mathbb{Z})m^2 - n^2 = a\Big)$.)

6. ***Prove*** that if a is an integer such that a is not of equal to $4k + 2$ for any integer k (that is, if the remainder of dividing a by 4 is 0 or 1 or 3 or, equivalently, if a is either odd or divisible by 4) then a can be written as the difference of two squares. (In formal language: $(\forall a \in \mathbb{Z})\Big((\exists k \in \mathbb{Z})(a = 4k \vee a = 4k + 1 \vee a = 4k + 3) \Longrightarrow (\exists m \in \mathbb{Z})(\exists n \in \mathbb{Z})m^2 - n^2 = a\Big)$.) HINT: Show that we can pick integers r, s such that $r - s$ is even $a = rs$, and $r \leq s$. (If a is odd just take $r = 1$ and $s = a$. If a is even take $r = 2$.) Then solve the equations $m - n = r$, $m + n = s$. ***Make sure you explain why this yields integers*** m, n ***such that*** $m^2 - n^2 = a$***. In particular, explain what goes wrong with this method for finding*** m, n ***in the "bad" case, when*** $a = 4k + 2$ ***for some integer*** k***.***