MATHEMATICS 300 — FALL 2018

Introduction to Mathematical Reasoning

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HOMEWORK ASSIGNMENT NO. 6, DUE ON THURSDAY, NOVEMBER 1 (FOR SECTION 5) AND FRIDAY, NOVEMBER 2 (FOR SECTION 3)

SOLUTIONS

Problem 1. For each of the following sentences in formal language,

1. ***Translate*** the sentence into ***reasonable*** English. (Please do not write horrors such as "if n an element of the set of natural numbers then n is a mamber of the set of even numbers or n is a member of the set of odd numbers". The way a normal English speaker would say that "a natural number is even or odd".)

2. ***Indicate*** whether the sentence is true or false.

3. ***Give a brief explanation*** (that is, a brief proof) of why the sentence is true or why it is false. (Please do not write long, incomprehensible horrors such as "the sentence $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})m > n$ is true because the set of natural numbers is infinite so that every natural number has other natural numbers that are associated to it where m is greater that n". Write instead: The sentence is true because given any $n \in \mathbb{N}$ we can take $m = n + 1$, and then $m > n$". This is short, precise, clear, and correct.)

1. $(\forall n \in \mathbb{Z})(\exists m \in \mathbb{Z})m < n$.

   Answer: For every integer n there exists an integer m such that $m < n$. This is ***true***, because given any $n \in \mathbb{Z}$ we can choose m to be $n-1$, and then $m < n$.

2. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})m < n$.

   Answer: For every natural number n there exists a natural number m such that $m < n$. This is ***false***, because if $n = 1$ then there is no $m \in \mathbb{N}$ such that $m < n$.

3. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})m \leq n$.

   Answer: For every natural number n there exists a natural number m such that $m \leq n$. This is ***true***, because given any $n \in \mathbb{N}$ we can choose m to be n, and then $m \leq n$.

4. $(\exists m \in \mathbb{N})(\forall n \in \mathbb{N})m \geq n$.

   Answer: There exists a natural number m such that m is greater than or equal to every natural number n. This is ***false*** because, given any $m \in \mathbb{N}$, m cannot be greater than or equal to every natural number n, since m is not greater than or equal to $m+1$.

5. $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(\forall z \in \mathbb{R})(z < y \Longrightarrow z < x)$.

   Answer: For every real number x there exists a real number y such that for every real number z if $z < y$ then $z < x$. This is ***true***, because given any $x \in \mathbb{R}$ we can choose y to be x, and then given any $z \in \mathbb{R}$ the implication "$z < y \Longrightarrow z < x$" says "$z < x \Longrightarrow z < x$", which is obviously true.

6. $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(\forall z \in \mathbb{R})(z < y \Longrightarrow x < z)$.

   Answer: For every real number x there exists a real number y such that for every real number z if $z < y$ then $x < z$. This is ***false***. To prove this it suffices to find one value of x for which the sentence

   $$(\exists y \in \mathbb{R})(\forall z \in \mathbb{R})(z < y \Longrightarrow x < z) \tag{0.1}$$

   is false. I will actually show much more than this: I will show that the sentence (0.1) is false for *every* $x \in \mathbb{R}$. Let $x \in \mathbb{R}$ be arbitarry. I will show that (0.1) is false. Suppose it was true. Then we could pick a witness, that is, a real number y for which the sentence

   $$(\forall z \in \mathbb{R})(z < y \Longrightarrow x < z) \tag{0.2}$$

   is true. This means that "$z < y \Longrightarrow x < z$" is true for every $z \in \mathbb{R}$. Let us pick[1] $z = -|x| - |y| - 1$. Then it follows from (0.2) by Rule $\forall_{use}$ that

   $$z < y \Longrightarrow x < z\,. \tag{0.3}$$

[1] Or we could pick $z = \min(x, y) - 1$.

On the other hand, since $z = -|x| - |y| - 1$, it follows that $z < x$ and $z < y$. So "$z < y$" is true, and "$x < z$" is false. Hence the implication "$z < y \Longrightarrow x < z$" is false. Therefore

$$\sim (z < y \Longrightarrow x < z)\,. \tag{0.4}$$

Therefore

$$(z < y \Longrightarrow x < z) \wedge \Big(\sim (z < y \Longrightarrow x < z)\Big)\,. \tag{0.5}$$

Clearly, (0.5) is a contradiction. This contradiction arose from assuming that (0.1) was true. Hence (0.1) is false.

7. $(\forall x \in \mathbb{R})(\forall z \in \mathbb{R})(\exists y \in \mathbb{R})(z < y \Longrightarrow x < z)$.

   Answer: For every real number x and every real number z there exists a real number y such that if $z < y$ then $x < z$. This is ***true***. Let x, z be arbitrary real numbers. Pick y to be z. Then "$z < y$" is false, so "$z < y \Longrightarrow x < z$" is true. So y is a witness for the sentence "$(\exists y \in \mathbb{R})(z < y \Longrightarrow x < z)$". So "$(\exists y \in \mathbb{R})(z < y \Longrightarrow x < z)$" is true.

8. $(\forall x \in \mathbb{R})(\forall z \in \mathbb{R})(z < x \Longrightarrow (\exists y \in \mathbb{R})(z < y < x)$.

   Answer: For every real number x and every real number z if $z < x$ then there exists a real number y such that $z < y < z$. This is ***true***. Let x, z be arbitrary real numbers. Assume that $z < x$. Pick $y = \frac{1}{2}(x + z)$. Then $z < y < x$. So y is a witness for "$(\exists y \in \mathbb{R})(z < y < x)$".

9. $(\forall x \in \mathbb{Z})(\forall z \in \mathbb{Z})(z < x \Longrightarrow (\exists y \in \mathbb{Z})(z < y < x)$.

   Answer: For every integer x and every integer z if $z < x$ then there exists an integer y such that $z < y < z$. This is ***false***. To see this, take $x = 3$, $z = 2$. Then "$z < x$" is true. But "$(\exists y \in \mathbb{R})(z < y < x)$" is false, because there is no integer y such that $2 < y < 3$.

10. $(\forall x \in \mathbb{Z})\Big(x > 0 \Longrightarrow (\exists y \in \mathbb{Z})(\forall z \in \mathbb{R})(z > y \Longrightarrow \frac{1}{z^2} < x)\Big)$.

Answer: For every integer x, if x is positive then there exists an integer y such that for every real number z if $z > y$ then $\frac{1}{z^2} < x$. This is ***true***. To see this, let x be an arbitrary integer. Assume $x > 0$. Pick $y = 1$. Then if z is an arbitrary integer the implication "$z > y \Longrightarrow \frac{1}{z^2} < x$ is true, because if $z > y$ then $z^2 > y^2$, so $\frac{1}{z^2} < \frac{1}{y^2} = 1$, while on the other hand $x \geq 1$, because $x \in \mathbb{Z}$ and $x > 0$, so $\frac{1}{z^2} < x$. Thus y is a witness for "$(\exists y \in \mathbb{Z})(\forall z \in \mathbb{R})(z > y \Longrightarrow \frac{1}{z^2} < x)$".

11. $(\forall X)\emptyset \in X$.

    Answer: The empty set belongs to every set. This is ***false***. To see this, take $X = \emptyset$. Then $\emptyset$ does not belong to X, because X is the empty set, so X has no members.

12. $(\forall X)\emptyset \subseteq X$.

    Answer: The empty set is a subset of every set. This is ***true***. This was proved in class.

13. $(\exists X)(\forall Y)X \in Y$.

    Answer: There exist sets X, Y such that X belongs to Y. This is ***true***. Just take $X = \emptyset$, $Y = \{\emptyset\}$.

14. $(\exists X)(\forall Y)X \subseteq Y$.

    Answer: There exists a set X that is a subset of every set. This is ***true***, because $\emptyset$ is a subset of every set,

15. $(\forall X)(\forall x)(x \in X \Longrightarrow \{x\} \in X)$.

    Answer: If X is a set and x is a member of X, then the singleton of x belongs to X. This is ***false***. Take X to be $\{\emptyset\}$ and take $x = \emptyset$. Then x belongs to X, but $\{x\}$ does not belong to X, since the only member of X is $\emptyset$, and $\{\emptyset\} \neq \emptyset$. (To see that $\{\emptyset\} \neq \emptyset$: $\{\emptyset\}$ has one member, and $\emptyset$ has no members, so they cannot be the same set.)

16. $(\forall X)(\forall x)(x \in X \Longrightarrow \{x\} \subseteq X)$.

    Answer: If X is a set and x is a member of X then the singleton of x is a subset of X. This is ***true***. To see that $\{x\} \subseteq X$ we have to show that every member of $\{x\}$ belongs to X. But the only member of $\{x\}$ is x, and x indeed belongs to X.

17. $(\forall X)(\forall Y)\Big(X \subseteq Y \Longrightarrow \{X\} \in \mathcal{P}(Y)\Big)$.

    Answer: If X and Y are sets and X is a subset of Y then the singleton of X belongs to the power set of Y. This is ***false***. Take $X = Y = \{\emptyset\}$. Then $X = Y$, so $X \subseteq Y$. But X is not a member of Y. (Reason: since $Y = \{\emptyset\}$, the only member of Y is $\emptyset$. But X is not $\emptyset$, because X is a singleton, so X has one member, whereas $\emptyset$ has no members. So X is not a member of Y.) Since $\mathcal{P}(Y)$ is the set of all subsets of Y, $\{X\}$ belongs to $\mathcal{P}(Y)$ if and only if $\{X\}$ is a subset of Y. And $\{X\}$ is a subset of Y if and only if every member of $\{X\}$ belongs to Y. Since X is the unique member of $\{X\}$, we see that $\{X\} \subseteq Y$ if and only if $X \in Y$. So $\{X\} \in \mathcal{P}(Y)$ if and only if $X \in Y$. But we have seen that $X \notin Y$. Hence $\{X\} \notin \mathcal{P}(Y)$.

18. $(\forall X)(\forall Y)(X \subseteq Y \Longrightarrow X \subseteq \mathcal{P}(Y))$.

    Answer: If X and Y are sets and X is a subset of Y then X is a subset of the power set of Y. This is ***false***. Take X to be the set $\{\,\{\emptyset\}\,\}$, and take $Y = X$. Then X a singleton, so X has only one member. And that unique member of X is $\{\emptyset\}$. But $\{\emptyset\}$ does not belong to $\mathcal{P}(Y)$). (*Proof:* Assume $\{\emptyset\}$ belongs to $\mathcal{P}(Y)$. Then $\{\emptyset\}$ is a subset of Y, i.e., of X. But this would imply that every member of $\{\emptyset\}$ belongs to X. Since $\{\emptyset\}$ has only one member, and that member is $\emptyset$, it would follow that $\emptyset$ belongs to X. But X has only one member, and that member is $\{\emptyset\}$. So $\{\emptyset\} = \emptyset$. But this is not possible, since $\{\emptyset\}$ has one member by $\emptyset$ has no members.) Since $\{\emptyset\} \in X$ but $\{\emptyset\} \notin \mathcal{P}(Y)$, it follows that X is not a subset of $\mathcal{P}(Y)$.

19. $(\forall X)(\forall Y)(X \subseteq Y \Longrightarrow \mathcal{P}(X) \subseteq \mathcal{P}(Y))$.

    Answer: If X and Y are sets X is a subset of Y then the power set of X is a subset of the power set of Y. This is ***true***. To prove this we have to show that every member of $\mathcal{P}(X)$ belongs to $\mathcal{P}(Y))$. So let Z be an arbitrary member of $\mathcal{P}(X)$. Then $Z \subseteq X$. Since $X \subseteq Y$, it follows that $Z \subseteq Y$, so $Z \in \mathcal{P}(Y)$.

Problem 2.

1. ***Find*** at least ten prime numbers p such that $p+4$ is also prime.

   Answer: 3, 7, 13, 19, 37, 43, 67, 79, 97, 103.

2. ***Prove*** that there exists a unique prime number p such that $p+4$ and $p+8$ are also prime, and ***find*** that number.

   Answer: Let $p \in \mathbb{Z}$ be such that p, $p+4$ and $p+8$ are prime, Using the division theorem, write $p = 3q + r$, with $q \in \mathbb{Z}$, $r \in \mathbb{Z}$, $0 \leq r < 3$. Then $r = 0 \vee r - 1 \vee r = 2$. We now prove that $r \neq 1$ and $r \neq 2$:

   - Assume $r = 1$. Then $p = 3q + 1$, so $p + 8 = 3q + 9$, and then $p + 8 = 3(q + 3)$. Hence $3|p + 8$. Since $p + 8$ is prime, $3 = p + 8$ or $3 = 1$. Since $3 \neq 1$, it follows that $3 = p + 8$, so $p = -5$, contradicting the fact that $p > 1$. So $r \neq 1$.
   - Next, assume $r = 2$. Then $p = 3q+2$, so $p+4 = 3q+6$, and then $p + 4 = 3(q + 2)$. Hence $3|p + 4$. Since $p + 4$ is prime, $3 = p + 4$ or $3 = 1$. Since $3 \neq 1$, it follows that $3 = p + 4$, so $p = -1$, contradicting the fact that $p > 1$. So $r \neq 2$.

   It follows that $r = 0$. Then $p = 3q$, and $q \in \mathbb{Z}$. So $3|p$. Since p is prime and $3|p$, either $3 = 1$ or $3 = p$. Since $3 \neq 1$, we find that $p = 3$.

   So the assumption that p, $p + 4$ and $p + 8$ are prime implies that $p = 3$.

   On the other hand, if $p = 3$ then $p + 4 = 7$ and $p + 8 = 11$, so p, $p + 4$ and $p + 8$ are prime. So 3 is the unique p such that p, $p + 4$ and $p + 8$ are prime.

3. ***Prove*** that there does not exist a prime number p such that $p+4$, $p+8$ and $p+12$ are also prime.

   Answer: Assume there does exist a prime number p such that $p+4$, $p+8$ and $p+12$ are also prime. Pick one. Since p, $p+4$, and $p+8$ prime, it follows from the previous part of the problem that $p=3$. Since $p+12$ is prime, and $p+12=15$, it follows that 15 is prime. But 15 is not prime, So we have proved a contradiction. Hence an integer p such that p, $p+4$, $p+8$ and $p+12$ are prime does not exist.

Problem 3. ***Prove*** the following statement:

(*) If n is an integer then $n(n+1)(n+2)(n+3)(n+4)$ is divisible by 120.

Answer:

Let $n \in \mathbb{Z}$ be arbitrary

Let $m = n(n+1)(n+2)(n+3)(n+4)$.

We are going to prove that

1. $8|m$,
2. $3|m$,
3. $5|m$,
4. $24|m$,
5. $120|m$.

Proof that $8|m$ *:*

Using the division theorem, write $n = 4q + r$, with $q \in \mathbb{Z}$, $r \in \mathbb{Z}$, $r = 0 \lor r = 1 \lor r = 2 \lor r = 3$.

Case 1: $r = 0$.

Then $n = 4q$.

So $n + 2 = 4q + 2$.

Hence

$$\begin{aligned} m &= n(n+1)(n+2)(n+3)(n+4) \\ &= 4q(n+1)(4q+2)(n+3)(n+4) \\ &= 8q(n+1)(2q+12)(n+3)(n+4)\,, \end{aligned}$$

so $\boxed{8|m}$.

Case 2: $r = 1$.

Then $n = 4q + 1$.

So $n + 3 = 4q + 4 = 4(q + 1)$.

And $n + 1 = 4q + 2 = 2(2q + 1)$.

Hence

$$\begin{aligned} m &= n(n+1)(n+2)(n+3)(n+4) \\ &= 2(2q+1)(n+1)4(q+1)(n+3)(n+4) \\ &= 8(2q+1)(n+1)(q+1)(n+3)(n+4)\,, \end{aligned}$$

so $\boxed{8|m}$.

Case 3: $r = 2$.

Then $n = 4q + 2 = 2(2q + 1)$.

And $n + 4 = 4q + 4 = 4(q + 1)$.

Hence

$$\begin{aligned} m &= n(n+1)(n+2)(n+3)(n+4) \\ &= 2(2q+1)(n+1)4(q+1)(n+3)(n+4) \\ &= 8(2q+1)(n+1)(q+1)(n+3)(n+4)\,, \end{aligned}$$

so $\boxed{8|m}$.

Case 4: $r = 3$.

Then $n = 4q + 3$.

So $n + 1 = 4q + 4 = 4(q + 1)$.

And $n + 3 = 4q + 6 = 2(2q + 3)$.

Hence

$$\begin{aligned} m &= n(n+1)(n+2)(n+3)(n+4) \\ &= 4n(q+1)(n+2)2(2q+3)(n+4) \\ &= 8n(q+1)(n+2)(2q+3)(n+4)\,, \end{aligned}$$

so $\boxed{8|m}$.

We have proved that $8|m$ in all four cases. So $\boxed{\boxed{8|m}}$.

Proof that $3|m$ *:*

Using the division theorem, write $n = 3q + r$, with $q \in \mathbb{Z}$, $r \in \mathbb{Z}$, $r = 0 \lor r = 1 \lor r = 2$.

Case 1: $r = 0$.

Then $n = 3q$.

So

$$\begin{aligned} m &= n(n+1)(n+2)(n+3)(n+4) \\ &= 3q(n+1)(n+2)(n+3)(n+4)\,, \end{aligned}$$

so $\boxed{3|m}$.

Case 2: $r = 1$.

Then $n = 3q + 1$.

So $n + 2 = 3q + 3 = 3(q + 1)$.

Hence

$$\begin{aligned} m &= n(n+1)(n+2)(n+3)(n+4) \\ &= n(n+1)(3q+3)(n+3)(n+4) \\ &= 3(q+1)n(n+1)(n+3)(n+4)\,, \end{aligned}$$

so $\boxed{3|m}$.

Case 3: $r = 2$.

Then $n + 1 = 3q + 3$.

So $n+1=3(q+1)$.

Hence

$$\begin{aligned} m &= n(n+1)(n+2)(n+3)(n+4) \\ &= 3n(q+1)(n+2)(n+3)(n+4)\,, \end{aligned}$$

so $\boxed{3|m}$.

We have proved that $3|m$ in all three cases. So $\boxed{\boxed{3|m}}$.

Proof that $5|m$ *:*

Using the division theorem, write $n=5q+r$, with $q\in\mathbb{Z}$, $r\in\mathbb{Z}$, $r=0\vee r=1\vee r=2\vee r=3\vee r=4$.

Case 1: $r=0$.

Then $n=5q$.

Hence

$$\begin{aligned} m &= n(n+1)(n+2)(n+3)(n+4) \\ &= 5q(n+1)(n+2)(n+3)(n+4)\,, \end{aligned}$$

so $\boxed{5|m}$.

Case 2: $r=1$.

Then $n+4=5q+5=5(q+1)$.

Hence

$$\begin{aligned} m &= n(n+1)(n+2)(n+3)(n+4) \\ &= 5n(n+1)(n+2)(n+3)(q+1))\,, \end{aligned}$$

so $\boxed{5|m}$.

Case 3: $r=2$.

Then $n+3=5q+5=5(q+1)$.

Hence

$$\begin{aligned} m &= n(n+1)(n+2)(n+3)(n+4) \\ &= 5n(n+1)(n+2)(q+1)(n+4)\,, \end{aligned}$$

so $\boxed{5|m}$.

Case 4: $r = 3$.

Then $n + 2 = 5q + 5 = 5(q + 1)$.

Hence

$$\begin{aligned} m &= n(n+1)(n+2)(n+3)(n+4) \\ &= 5n(n+1)(q+1)(n+3)(n+4)\,, \end{aligned}$$

so $\boxed{5|m}$.

Case 5: $r = 4$.

Then $n + 1 = 5q + 5 = 5(q + 1)$.

Hence

$$\begin{aligned} m &= n(n+1)(n+2)(n+3)(n+4) \\ &= 5n(q+1)(n+2)(n+3)(n+4)\,, \end{aligned}$$

so $\boxed{5|m}$.

We have proved that $5|m$ in all five cases. So $\boxed{\boxed{6|m}}$.

So we have proved that $8|m$, $3|m$, and $5|m$.

We now prove that $24|m$.

Since $3|m$, $8|m$, and $5|m$, we may write

$$m = 3j\,, \quad m = 8k\,, \quad \text{and} \quad m = 5\ell\,, \quad \text{with} \quad j, k, \ell \in \mathbb{Z}\,.$$

On the other hand,

$$1 = 16 - 15\,.$$

Therefore

$$\begin{aligned} m &= 16m - 15m \\ &= 16 \times 3j - 15 \times 8k \\ &= (8 \times 3)2j - (3 \times 8) \times 5k \\ &= 24(2j - 5k)\,. \end{aligned}$$

So $\boxed{\boxed{24|m}}$.

Finally, we prove that $120|m$.

We have

$$1 = 25 - 24\,.$$

Therefore

$$\begin{aligned} m &= 25m - 24m \\ &= 25 \times 24(2j - 5k) - 24 \times 5\ell \\ &= 5 \times 24 \times 5(2j - 5k) - (24 \times 5)\ell \\ &= 120 \times 5(2j - 5k) - 120\ell \\ &= 120\Big((2j - 5k) - \ell\Big)\,. \end{aligned}$$

So $\boxed{\boxed{120|m}}$. **Q.E.D.**