MATHEMATICS 300 — FALL 2018

Introduction to Mathematical Reasoning

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HOMEWORK ASSIGNMENT NO. 6, DUE ON THURSDAY, NOVEMBER 1 (FOR SECTION 5) AND FRIDAY, NOVEMBER 2 (FOR SECTION 3)

Problem 1. For each of the following sentences in formal language,

1. ***Translate*** the sentence into ***reasonable*** English. (Please do not write horrors such as "if n an element of the set of natural numbers then n is a mamber of the set of even numbers or n is a member of the set of odd numbers". The way a normal English speaker would say that "a natural number is even or odd".)
2. ***Indicate*** whether the sentence is true or false.
3. ***Gicve a brief explanation*** (that is, a brief proof) of why the sentence is true or why it is false. (Please do not write long, incomprehensible horrors such as "the sentence $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})m > n$ is true because the set of natural numbers is infinite so that every natural number has other natural numbers that are associated to it where m is greater that n". Write instead: The sentence is true because givden any $n \in \mathbb{N}$ we can take $m = n + 1$, and then $m > n$". This is short, precise, clear, and correct.)

1. $(\forall n \in \mathbb{Z})(\exists m \in \mathbb{Z})m < n$.
2. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})m < n$.
3. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})m \leq n$.
4. $(\exists m \in \mathbb{N})(\forall n \in \mathbb{N})m \geq n$.
5. $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(\forall z \in \mathbb{R})(z < y \Longrightarrow z < x)$.
6. $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(\forall z \in \mathbb{R})(z < y \Longrightarrow x < z)$.
7. $(\forall x \in \mathbb{R})(\forall z \in \mathbb{R})(\exists y \in \mathbb{R})(z < y \Longrightarrow x < z)$.
8. $(\forall x \in \mathbb{R})(\forall z \in \mathbb{R})(z < x \Longrightarrow (\exists y \in \mathbb{R})(z < y < x)$.
9. $(\forall x \in \mathbb{Z})(\forall z \in \mathbb{Z})(z < x \Longrightarrow (\exists y \in \mathbb{Z})(z < y < x)$.
10. $(\forall x \in \mathbb{Z})\Big(x > 0 \Longrightarrow (\exists y \in \mathbb{Z})(\forall z \in \mathbb{R})(z > y \Longrightarrow \frac{1}{z^2} < x)\Big)$. (NOTE: This sentence is actually the mathematically precise way to say "$\lim_{x\to\infty} \frac{1}{x^2} = 0$". Actually, all of Calculus is really about complicated sentences involving quantifiers.)

11. $(\forall X)\emptyset \in X$.
12. $(\forall X)\emptyset \subseteq X$.
13. $(\exists X)(\forall Y)X \in Y$.
14. $(\exists X)(\forall Y)X \subseteq Y$.
15. $(\forall X)(\forall x)(x \in X \Longrightarrow \{x\} \in X)$.
16. $(\forall X)(\forall x)(x \in X \Longrightarrow \{x\} \subseteq X)$.
17. $(\forall X)(\forall Y)\Big(X \subseteq Y \Longrightarrow \{X\} \in \mathcal{P}(Y)\Big)$. (NOTE: If X is a set then $\mathcal{P}(X)$ is the set of all subsets of X; That is, $\mathcal{P}(X) = \{Y : Y \subseteq X\}$.
18. $(\forall X)(\forall Y)(X \subseteq Y \Longrightarrow X \subseteq \mathcal{P}(Y))$.
19. $(\forall X)(\forall Y)(X \subseteq Y \Longrightarrow \mathcal{P}(X) \subseteq \mathcal{P}(Y))$.

(HINT: Study carefully the answers I wrote for the problems in the first midterm exam.)

Problem 2.

1. ***Find*** at least ten prime numbers p such that $p + 4$ is also prime.

2. ***Prove*** that there exists a unique prime number p such that $p + 4$ and $p + 8$ are also prime, and ***find*** that number.

3. ***Prove*** that there does not exist a prime number p such that $p + 4$, $p + 8$ and $p + 12$ are also prime.

(HINT: Study carefully the answers I wrote for the problems in the first midterm exam.)

Problem 3. ***Prove*** the following statement:

(*) If n is an integer then $n(n+1)(n+2)(n+3)(n+4)$ is divisible by 120.

(HINT: Study carefully the answers I wrote for the problems in the first midterm exam.)

You are allowed to do this problem by induction, although I do not recommend it. But remember that induction is, in principle, a method for proving statements for all natural numbers n. The statement you are asked to prove here is for all integers n, so if you do it by induction you still need another argument to conclude, after you have proved the statement for all $n \in \mathbb{N}$, that it is true for all $n \in \mathbb{Z}$.