MATHEMATICS 300 — FALL 2018

Introduction to Mathematical Reasoning

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HOMEWORK ASSIGNMENT NO. 7, DUE ON THURSDAY, NOVEMBER 8 (FOR SECTION 5) AND FRIDAY, NOVEMBER 9 (FOR SECTION 3)

Problem 1. ***Translate*** into reasonable English and ***prove*** the following sentences:

1. $(\forall x \in \mathbb{R})\Big(x > 0 \Longrightarrow (\exists y \in \mathbb{R})(\forall z \in \mathbb{R})(z > y \Longrightarrow \frac{1}{z^2} < x)\Big)$.
   (NOTE: This sentence is actually the mathematically precise way to say "$\lim_{x\to\infty} \frac{1}{x^2} = 0$".)
2. $$(\forall x \in \mathbb{R})\Bigg(x > 0 \Longrightarrow (\exists y \in \mathbb{R}) \Big(y > 0 \wedge (\forall z \in \mathbb{R})(0 < |z-3| < y \Longrightarrow |z^2 - 9| < x)\Big)\Bigg).$$
   (NOTE: This sentence is actually the mathematically precise way to say "$\lim_{x\to 3} x^2 = 9$".)

NOTES: The first sentence was supposed to be Sentence No. 10 of Problem 1 of Homework assignment No. 6, but came out different because of a couple of typos.

As this problem shows,

all of analysis (that is, almost all of mathematics) is really about complicated sentences involving lots of quantifiers. So if you want to do mathematics you have to learn to work with quantifiers.

Problem 2. Definition. The power set of a set S is the set $\mathcal{P}(S)$ given by

$$\mathcal{P}(S) = \{X : X \subseteq S\}.$$

(That is, $\mathcal{P}(S)$ is the set of all subsets of S.) □

(I) ***Prove or disprove*** each of the following sentences:

1. $(\forall X)\,\emptyset \in \mathcal{P}(X)$.
2. $(\forall X)\,\emptyset \subseteq \mathcal{P}(X)$.
3. $(\forall X)(\forall Y)(\mathcal{P}(X) \subseteq \mathcal{P}(Y) \iff X \subseteq Y)$.
4. $(\forall X)(\forall Y)\Big(\mathcal{P}(X \cup Y) = \mathcal{P}(X) \cup \mathcal{P}(Y)\Big)$.
5. $(\forall X)(\forall Y)\Big(\mathcal{P}(X \cap Y) = \mathcal{P}(X) \cap \mathcal{P}(Y)\Big)$.

(II) ***Give a grade***, on a scale from 0 to 5, to each of the following definitions of "power set", and ***explain*** the reason for your grade. You are also allowed to give a grade of -10 (a.k.a "horrendous"). NOTE: Almost every one of the definitions in the list has something wrong, so very few of your grades should be 5. But there is at least one definition that is perfectly correct and deserves a 5.

1. The power set of a set is the set $\mathcal{P}(S)$ given by
$$\mathcal{P}(S) = \{X : X \subseteq S\}\,.$$
2. The power set of a set S contains all the subsets of S.
3. The power set of a set S contains all the subsets of S.
4. The power set of a set S consists of all the subsets of S.
5. Power set is when you form all the sets that are part of a set.
6. Suppose you form a set Q as follows: you pick a set S, and declare that the members of the set Q are all the subsets of S. Then the set Q is called the power set of S.

Problem 3. (This problem is about ***partitions***. To understand it, you should first read the explanation of "partitions" below.)

1. ***Prove*** Theorem I below.
2. *(In this list of questions, if A is a set then "$\mathcal{P}(A)$" stands for "the power set of A", that is, the set $\{X : X \subseteq A\}$. The meaning of "$\mathbf{P}_b$" is explained in the section on partitions below.)* ***Indicate*** which of the following are true and which are false (if $b \in \mathbb{N}$):

(a) $\mathbf{P}_b \in \mathbb{Z}$,

(b) $\mathbf{P}_b \subseteq \mathbb{Z}$,

(c) $\mathbf{P}_b \in \mathcal{P}(\mathbb{Z})$,

(d) $\mathbf{P}_b \subseteq \mathcal{P}(\mathbb{Z})$,

(e) $\mathbf{P}_b \in \mathcal{P}\Big(\mathcal{P}(\mathbb{Z})\Big)$,

(f) $\mathbf{P}_b \subseteq \mathcal{P}\Big(\mathcal{P}(\mathbb{Z})\Big)$,

(g) $\Big\{\mathbf{P}_b\Big\} \in \mathcal{P}\Big(\mathcal{P}(\mathbb{Z})\Big)$,

(h) $\Big\{\mathbf{P}_b\Big\} \subseteq \mathcal{P}\Big(\mathcal{P}(\mathbb{Z})\Big)$,

(i) $\Big\{\mathbf{P}_b\Big\} \in \mathcal{P}\Bigg(\mathcal{P}\Big(\mathcal{P}(\mathbb{Z})\Big)\Bigg)$,

(j) $\mathbf{P}_b$ has b members,

(k) $\mathbf{P}_b = b$,

(l) $\mathbf{P}_b$ is an infinite set,

(m) every member of $\mathbf{P}_b$ is an infinite set.

3. If $b_1 \in \mathbb{N}$, $b_2 \in \mathbb{N}$, then the condition that b_1 divides b_2 (i.e., that $b_1|b_2$) is equivalent to a condition C_{b_1,b_2} about the partitions $\mathbf{P}_{b_1}$. $\mathbf{P}_{b_2}$. (This means "for all $b_1, b_2 \in \mathbb{N}$, $b_1|b_2 \iff C_{b_1,b_2}$".) ***Indicate*** which of the following is the condition C_{b_1,b_2}, and ***sketch the proof*** that the condition you chose is equivalent to "$b_1|b_2$".

   (a) $\mathbf{P}_{b_1} \subseteq \mathbf{P}_{b_2}$,

   (b) $\mathbf{P}_{b_2} \subseteq \mathbf{P}_{b_1}$,

   (c) $(\forall X)(X \in \mathbf{P}_{b_1} \Longrightarrow X \in \mathbf{P}_{b_2})$.

   (d) $(\forall X)(X \in \mathbf{P}_{b_2} \Longrightarrow X \in \mathbf{P}_{b_1})$.

   (e) $(\forall X)\Big(X \in \mathbf{P}_{b_1} \Longrightarrow (\exists Y \in \mathbf{P}_{b_2})X \subseteq Y\Big)$,

   (f) $(\forall X)\Big(X \in \mathbf{P}_{b_2} \Longrightarrow (\exists Y \in \mathbf{P}_{b_1})X \subseteq Y\Big)$.

   (HINT: Think of an example.)

Partitions

If you are asked to define "partition", the first two questions that you have to ask yourself is this: ***is "partition" a predicate or a term?***, and ***how many arguments does it have, and what kinds of things are they?***
The answer is: "partition" is a ***2-argument predicate***: we say things like "$\mathbf{P}$ is a partition of A". And both arguments are ***sets***, that is, we talk about a set $\mathbf{P}$ being a partition of a set A. And, furthermore, the first argument $\mathbf{P}$ must be a set of subsets of the second argument A, i.e., a set whose members are subsets of A.

And now that we know what kind of thing a "partition" is, we can write the definition:

Definition. A partition of a set S is a set $\mathbf{P}$ such that:

1. Every member of $\mathbf{P}$ is a nonempty subset of S.
2. If X, Y are members of $\mathbf{P}$ and $X \neq Y$ then $X \cap Y = \emptyset$.
3. If s is an arbitrary member of S, then $s \in X$ for some member X of $\mathbf{P}$. $\square$

(In almost fully formal language:

$\mathbf{P}$ is a partition of S if and only if

$$\begin{aligned}&(\forall x)\Big(x \in \mathbf{P} \Longrightarrow (x \subseteq S \wedge x \neq \emptyset)\Big)\\ &\quad\wedge(\forall X)(\forall Y)\Big((X \in \mathbf{P} \wedge Y \in \mathbf{P}) \Longrightarrow (X = Y \vee X \cap Y = \emptyset)\Big)\\ &\quad\wedge(\forall s \in S)(\exists X \in \mathbf{P})s \in X\,.\end{aligned}$$

If you want to write it in 100% fully formal language, then you need a symbolic notation for "$\mathbf{P}$ is a partition of S". You could agree to write

"Part($\mathbf{P}, S$) for "$\mathbf{P}$ is a partition of S", and then the definition of "partition", in fully formal language, beocmes

$$(\forall \mathbf{P})(\forall S)\Bigg[\text{Part}(\mathbf{P}, S) \Longleftrightarrow$$

$$\Bigg((\forall x)\Big(x \in \mathbf{P} \Longrightarrow (x \subseteq S \wedge x \neq \emptyset)\Big)$$

$$\wedge(\forall X)(\forall Y)\Big((X \in \mathbf{P} \wedge Y \in \mathbf{P}) \Longrightarrow (X = Y \vee X \cap Y = \emptyset)\Big)$$

$$\wedge(\forall s \in S)(\exists X \in \mathbf{P}) s \in X\Bigg)\Bigg].$$

Example 1. Let S be the set of all Math 300 students this semester. Then we can construct a partition of S by dividing S into the math 300 sections: let A be the set of all Math sections; for each $a \in A$ (that is, for each section a) let T_a be the set of all the students who are in section a. Let $\mathbf{P}$ be the set whose members are all the sets T_a, for all the sections a. (That is, $\mathbf{P} = \{X : (\exists a \in A)X = T_a\}$.) Then it is clear that $\mathbf{P}$ is a partition of S.

If it is not clear to you how to prove that $\mathbf{P}$ is a partition of S, then ***prove*** it, and when you do that it will become clear. To prove that $\mathbf{P}$ is a parition of S, you have to prove: (a) for every $a \in A$, $T_a \neq \emptyset$ (HINT: if T_a was $\emptyset$, section a would be running, because we are not allowed to run a section with zero students); (b) If $X, Y \in \mathbf{P}$, either $X = Y$ or $X \cap Y = \emptyset$ (HINT: $X = T_a$ and $Y = T_b$ for some $a, b \in A$; if $a = b$ then $X = Y4; if4a \neq b$ then $X \cap Y = \emptyset$ because no student is registered in two different sections); (c) if $s \in S$ then $s \in X$ for some $X \in \mathbf{P}$ (HINT: if s is a Math 300 student then s must be registered in some section.))

Example 2. Let A be the set of all people who live in the United States. Let us pretend, to make this example easier, that all the people who live in the U.S. live in one of the 50 states. (That is, we pretend that here are no territories such as Puerto Rico or Guam, which are part of the U.S. but are not in one of the states.) For each state s, let X_s be the set of all people who live in s. (For example: $X_{\text{New Jersey}}$ is the set of all the people who live in New Jersey; X_{Alabama} is the set of all the people who live in Alabama; and so on.)

Let $\mathbf{P}$ be the set whose members are the 50 sets X_s. That is, if we let S

be the set whose members are the 50 states:

$$\mathbf{P} = \{x : (\exists s \in S) x = X_s\}\,.$$

Then $\mathbf{P}$ ***is a partition of*** A. (You should make sure that you understand how to prove this, although I am not asking you to hand in the proof.)

Example 3. Let $\mathcal{E}$ be the set of all even integers, and let $\mathcal{O}$ be the set of all odd integers. That is,

$$\begin{aligned} \mathcal{E} &= \{n \in \mathbb{Z} : 2|n\}\,, \\ \mathcal{O} &= \{n \in \mathbb{Z} : 2|n-1\}\,. \end{aligned}$$

Let

$$\mathbf{P} = \{\mathcal{E}, \mathcal{O}\}\,,$$

so $\mathbf{P}$ is the two-member set whose members are the sets $\mathcal{E}$ and $\mathcal{O}$.

Then $\mathbf{P}$ ***is a partition of*** $\mathbb{Z}$.

Proof: We have to prove that

1. Every member of $\mathbf{P}$ is a nonempty subset of $\mathbb{Z}$.
2. If X, Y are members of $\mathbf{P}$ and $X \neq Y$ then $X \cap Y = \emptyset$.
3. If n is an arbitrary member of $\mathbb{Z}$, then $n \in X$ for some member X of $\mathbf{P}$.

Condition 1 is easy: There are two members of $\mathbf{P}$, namely, $\mathcal{E}$ and $\mathcal{O}$, and they are both nonempty: $\mathcal{E}$ is nonempty because, for example, $2 \in \mathcal{E}$, and $\mathcal{O}$ is nonempty because, for example, $1 \in \mathcal{O}$.

To prove Condition 2 we have to show that $\mathcal{E} \cap \mathcal{O} = \emptyset$. But that follows from the fact that no integer can be both even and odd, which tells us precisely that no integer n can belong to both $\mathcal{E}$ and $\mathcal{O}$, that is, that no integer n can belong to $\mathcal{E} \cap \mathcal{O}$.

To prove Condition 3 we have to show that $\mathcal{E} \cup \mathcal{O} = \mathbb{Z}$. But that follows from the fact that every integer is even or odd, which tells us precisely that every integer n belongs to $\mathcal{E}$ or to $\mathcal{O}$, that is, every integer belongs to $\mathcal{E} \cup \mathcal{O}$, so $\mathcal{E} \cup \mathcal{O} = \mathbb{Z}$.

Example 4. For each natural number b, and each integer r such that $0 \leq r < b$, define a subset $E_{b,r}$ of $\mathbb{Z}$ by letting

$$E_{b,r} = \{n \in \mathbb{Z} : b|n - r\} .$$

In other words, $E_{b,r}$ is the set of all integers n such that the remainder of dividing n by b is r. So, for example,

- $E_{3,0}$ is the set of all integers that are divisible by 3. So the members of $E_{3,0}$ are $0, 3, -3, 6, -6, 9, -9$ and so on.
- $E_{3,1}$ is the set of all integers n such that the remainder of dividing n by 3 is 1. So the members of $E_{3,1}$ are $1, 4, -2, 7, -5, 10, -8$ and so on.
- $E_{3,2}$ is the set of all integers n such that the remainder of dividing n by 3 is 2. So the members of $E_{3,2}$ are $2, 5, -1, 8, -4, 11, -7$ and so on.

For each natural number b, we define a set $\mathbf{P}_b$ of subsets of $\mathbb{Z}$ as follows: $\mathbf{P}_b$ is the set whose members are the sets $E_{b,0}, E_{b,1}, \ldots, E_{b,b-1}$. In other words:

$$\mathbf{P}_b = \{X : (\exists r \in \mathbb{Z})(0 \leq r < b \wedge X = E_{b,r}\} .$$

So, for example:

$$\begin{aligned} \mathbf{P}_2 &= \Big\{ \mathcal{E}, \mathcal{O} \Big\} , \\ \mathbf{P}_3 &= \Big\{ E_{3,0}, E_{3,1}, E_{3,2} \Big\} , \\ \mathbf{P}_4 &= \Big\{ E_{4,0}, E_{4,1}, E_{4,2}, E_{4,3} \Big\} . \end{aligned}$$

Theorem I. For every natural number b, the set $\mathbf{P}_b$ is a partition of $\mathbb{Z}$.

Proof. **YOU DO IT.**