MATHEMATICS 300 — FALL 2018

Introduction to Mathematical Reasoning

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HOMEWORK ASSIGNMENT NO. 8, DUE ON THURSDAY, NOVEMBER 15 (FOR SECTION 5) AND FRIDAY, NOVEMBER 16 (FOR SECTION 3)

IMPORTANT REMARK. *The definition of "inductive set" given in the book is slightly different from the one I gave in class:*

- *The book's definition says: a subset S of $\mathbb{N}$ is inductive if*

$$(\forall n \in \mathbb{N})(n \in S \Longrightarrow n+1 \in S)\,.$$

- *My definition says: a subset S of $\mathbb{N}$ is inductive if*

$$1 \in S \wedge (\forall n \in \mathbb{N})(n \in S \Longrightarrow n+1 \in S)\,.$$

In other words: a subset S is inductive in my sense if and only if it is inductive in the book's sense and in addition $1 \in S$.

This homework assignment is about induction. The only way you will learn induction is by doing lots and lots and lots of problems. The book has a lot of problems, but I cannot assign as many problems as I should because when I do that students complain that it is too much work. But you cannot have it both ways: either you do a lot of work or you will not learn much.

I am only assigning a few problems. But I strongly recommend that you practise a lot, by doing ***all*** *the problems in sections 2.4 and 2.5. Several of these problems have answers at the end of the book, so I especially recommend those problems. because you can do them and then compare your answer with the book's anser.*

Problem 1. Book, pages 123-124, Problem 4, parts (d), (e), (k), (l).

Problem 2. Book, page 124, Problem 5, parts (d), (h), (j), (n), (q).

Problem 3. Book, page 126, Problem 10. *This problem requires some writing. Please write clearly. Explain how you would move the disks in a clear, precise, understandable way. Make sure you explain how you get the number $2^n - 1$.*

Problem 4. Book, pages 126-127, Problem 12, Parts (c) and (d).

Problem 5. For each of the following two theorems and alleged proofs:

1. ***Indicate*** whether the proof is correct.
2. ***Indicate*** whether the theorem is true.
3. If the theorem is false (in which case the alleged proof ***must*** be wrong), then
   (a) ***Find*** the mistake or mistakes in the proof. (Do not mention small mistakes such as "the third step is not justified", or "the sixth step is not clear". I amonly interested is serious mistakes, that make the proof invalid.)
   (b) ***Prove*** that the theorem is false.
4. If the theorem is true but the proof is invalid, then
   (a) ***Explain*** what is wrong with the proof.
   (b) ***Give*** a correct proof.
5. If the theorem and the proof are correct, all you need to do is say so.

Theorem I. If n is a natural number, then $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Proof:

- For $n = 1$, $1+2+\cdots+n = 1$ and $\frac{n(n+1)}{2} = 1$, so $1+2+\cdots+n = \frac{n(n+1)}{2}$.
- For $n = 2$, $1+2+\cdots+n = 3$ and $\frac{n(n+1)}{2} = 3$, so $1+2+\cdots+n = \frac{n(n+1)}{2}$.
- For $n = 3$, $1+2+\cdots+n = 6$ and $\frac{n(n+1)}{2} = 6$, so $1+2+\cdots+n = \frac{n(n+1)}{2}$.
- For $n = 4$, $1+2+\cdots+n = 10$ and $\frac{n(n+1)}{2} = 10$, so $1+2+\cdots+n = \frac{n(n+1)}{2}$.
- For $n = 5$, $1+2+\cdots+n = 15$ and $\frac{n(n+1)}{2} = 15$, so $1+2+\cdots+n = \frac{n(n+1)}{2}$.
- For $n = 6$, $1+2+\cdots+n = 21$ and $\frac{n(n+1)}{2} = 21$, so $1+2+\cdots+n = \frac{n(n+1)}{2}$.
- For $n = 7$, $1+2+\cdots+n = 28$ and $\frac{n(n+1)}{2} = 28$, so $1+2+\cdots+n = \frac{n(n+1)}{2}$.
- For $n = 8$, $1+2+\cdots+n = 36$ and $\frac{n(n+1)}{2} = 36$, so $1+2+\cdots+n = \frac{n(n+1)}{2}$.

so we see that $1 + +2 \cdots + n = \frac{n(n+1)}{2}$ for every $n \in \mathbb{N}$.

Theorem II. If n is a natural number, then $n^2 + n + 41$ is a prime number.

Proof:

- For $n = 1$, $n^2 + n + 41 = 43$, which is prime.
- For $n = 2$, $n^2 + n + 41 = 47$, which is prime.
- For $n = 3$, $n^2 + n + 41 = 53$, which is prime.
- For $n = 4$, $n^2 + n + 41 = 61$, which is prime.
- For $n = 5$, $n^2 + n + 41 = 71$, which is prime.
- For $n = 6$, $n^2 + n + 41 = 83$, which is prime.
- For $n = 7$, $n^2 + n + 41 = 97$, which is prime.
- For $n = 8$, $n^2 + n + 41 = 97$, which is prime.
- For $n = 9$, $n^2 + n + 41 = 131$, which is prime.
- For $n = 10$, $n^2 + n + 41 = 151$, which is prime.
- For $n = 11$, $n^2 + n + 41 = 173$, which is prime.
- For $n = 12$, $n^2 + n + 41 = 197$, which is prime.

So we see that $n^2 + n + 41$ is prime for every $n \in \mathbb{N}$.

NOTE: Actually, you can go on with this computation for $n = 13$, $n = 14$, $n = 15$, *and so on, all the way until* $n = 39$. *And, amazingly,* ***you always get a prime number !!!*** *For example, for* $n = 30$, $n^2+n+41 = 971$, *which is prime.*

NOTE: Do not forget the story of the turkey who sees that every day Mr. Bob comes to bring food, so the turkey draws the general conclusion (universal sentence) that

$$(\forall n) \text{ on Day } n \text{ Mr. Bob comes to feed me.}$$

Until one day, a few days before Thanksgiving, Mr. Bob comes to visit the turkey, but with a different purpose in mind.