MATHEMATICS 300 — FALL 2019

Introduction to Mathematical Reasoning

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HOMEWORK ASSIGNMENT NO. 7, DUE ON WEDNESDAY, NOVEMBER 20

This assignment consists of 9 problems.

The Fibonacci numbers f_n (for[1] $n \in \mathbb{N} \cup \{0\}$) are defined as follows:

$$\begin{aligned} f_0 &= 0\,, \\ f_1 &= 1\,, \\ f_n &= f_{n-1} + f_{n-2} \qquad \text{for } n \in \mathbb{N}\,,\ n > 1\,. \end{aligned}$$

Remark 1. The definition of the Fibonacci numbers looks very much like an inductive definition, except that, instead of defining each Fibonacci number in terms of the previous one, we define each Fibonacci number in terms of the ***two*** preceding ones. For this reason, the definition of the Fibonacci numbers is said to be a ***two-step inductive definition***. □

Example 1. Here are the first twelve Fibonacci numbers:

$$\begin{array}{llll} f_0 = 0\,, & f_1 = 1\,, & f_2 = 1\,, & f_3 = 2\,, \\ f_4 = 3\,, & f_5 = 5\,, & f_6 = 8\,, & f_7 = 13\,, \\ f_8 = 21\,, & f_9 = 34\,, & f_{10} = 55\,, & f_{11} = 89\,. \end{array}$$

I will now ask you to prove several facts about the Fibonacci numbers. In some of these results, there appears a very famous number, the "golden ratio". So we first define this number.

Definition 1. The golden ratio is the real number φ given by

$$\varphi = \frac{1+\sqrt{5}}{2}\,.$$

[1]Recall that "$\mathbb{N} \cup \{0\}$" is the set consisting of all the natural numbers and zero, so "$n \in \mathbb{N} \cup \{0\}$" means "$n \in \mathbb{N} \vee n = 0$".

Remark 2. The golden ratio also has several other names: the golden mean, the golden section, the divine proportion, the divine section, and also the golden proportion.

If you want to find out why this number is so important and so famous, you should read something about it:

Strongly recommended reading: The Wikipedia article entitled "golden ratio".

If you look at the first 12 Fibonacci numbers, it appears that they follow a pattern: even-odd-odd, even-odd-odd, etc. This is made precise in the follwoing theorem:

Theorem 1. *For every $n \in \mathbb{N} \cup \{0\}$, f_{3n} is even, and f_{3n+1} and f_{3n+2} are odd.*

Problem 1. ***Prove*** Theorem 1.
HINT: Use induction. □

Problem 2. ***Prove*** the identity

$$\sum_{k=1}^{n} f_k = f_{n+2} - 1 \text{ for } n \in \mathbb{N}. \tag{1}$$

HINT: Use induction. □

Problem 3. ***Prove*** the identity

$$\sum_{k=0}^{n-1} f_{2k+1} = f_{2n} \text{ for } n \in \mathbb{N}. \tag{2}$$

HINT: Use induction. □

Problem 4. ***Prove*** the identity

$$\sum_{k=1}^{n} f_{2k} = f_{2n+1} - 1 \text{ for } n \in \mathbb{N}. \tag{3}$$

HINT: Use induction. □

Problem 5. ***Prove*** the identity

$$\sum_{k=1}^{n} f_k^2 = f_n f_{n+1} \ \text{ for } \ n \in \mathbb{N}\,. \tag{4}$$

HINT: Use induction. □

Theorem 2. *(Binet's formula) The Fibonacci numbers f_n satisfy the identity*

$$f_n = \frac{\varphi^n - \psi^n}{\sqrt{5}} \qquad \text{for every} \quad n \in \mathbb{N}\,, \tag{5}$$

where φ is the golden ratio, that is, $\varphi = \frac{1+\sqrt{5}}{2}$, and ψ is the number given by

$$\psi = \frac{1-\sqrt{5}}{2}\,.$$

Problem 6. Prove Theorem 2.
HINTS:

1. Use strong induction.

2. Prove first that the numbers φ and ψ are solutions of the equation

$$x^2 = 1 + x\,,$$

i.e., that

$$\varphi^2 = 1 + \varphi\,.$$

and

$$\psi^2 = 1 + \psi\,.$$

Problem 7. ***The purpose of the result of this problem is to provide an easy way to compute approximately the Fibonacci numbers f_n for large n.***

1. ***Prove*** that $\left|\frac{\psi^n}{\sqrt{5}}\right| < 0.5$ for every $n \in \mathbb{N}$. (Actually, for large n $\left|\frac{\psi^n}{\sqrt{5}}\right| <$ 0.5 is much smaller that 0.5, but I am not asking you to do that.

2. ***Conclude*** from the previous result that f_n is the integer closest to $\frac{\varphi^n}{\sqrt{5}}$.

Theorem 3. *The powers φ^n of the golden ratio φ and the Fibonacci numbers f_n satisfy the identity*

$$\varphi^n = f_n\varphi + f_{n-1} \text{ for } n \in \mathbb{N}\,. \tag{6}$$

Problem 8. Prove Theorem 3. □

A 2 × 2 matrix is a square array

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where a, b, c, d are real numbers.

The sum $M_1 + M_2$ and the product $M_1 \cdot M_2$ of two 2×2 matrices $M_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $M_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$, are defined by the formulas

$$M_1 + M_2 = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}, \quad M_1 \cdot M_2 = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}.$$

If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant of M is the number $\det(M)$ given by

$$\det(M) = ad - bc\,.$$

The n-th power M^n of a 2×2 matrix M is defined inductively like the powers of a real number:

$$\begin{aligned} M^1 &= M\,, \\ M^{n+1} &= M^n \cdot M\,, \quad \text{for } n \in \mathbb{N}\,. \end{aligned}$$

Problem 9. ***Prove*** the ***Cassini identity***,

$$f_n^2 - f_{n+1}f_{n-1} = (-1)^{n-1} \text{ for } n \in \mathbb{N}\,,$$

by doing the following:

1. First, ***prove*** that if M, N are 2×2 matrices, then

$$\det(M \cdot N) = \det(M) \cdot \det(N)\,.$$

2. ***Conclude*** from the previous step that

$$\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n\right) = (-1)^n \text{ for } n \in \mathbb{N}\,.$$

3. ***Prove*** by induction that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} \text{ for } n \in \mathbb{N}\,. \tag{7}$$

4. ***Compute*** the determinant of the right-hand side of Equation (7).