

MATHEMATICS 300 — FALL 2017

Introduction to Mathematical Reasoning

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INSTRUCTOR'S NOTES

PART VI

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15 Operations on sets

There are several operations that enable us to construct new sets from given sets.

15.1 The power set of a set

Definition 31 Let A be a set. The power set of A is the set $\mathcal{P}(A)$ given by

$$\mathcal{P}(A) = \{X : X \subseteq A\}.$$

In other words, $\mathcal{P}(A)$ (*read as “the power set of A ”*) *is the set whose members are all the subsets of A .*

The *membership criterion* for the power set $\mathcal{P}(A)$ is the sentence “ $X \subseteq A$ ”. That is, for an object X to qualify as a member of $\mathcal{P}(A)$, it has to be shown that X is a subset of A .

Example 53. If $A = \{1, 2, 3\}$ then

$$\mathcal{P}(A) = \left\{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \right\}.$$

Notice that A is a finite set with 3 members, and $\mathcal{P}(A)$ has turned out to be a finite set with 8 members. ***This is not a coincidence. We will prove later that: if A is a finite set and A has n members, then the power set $\mathcal{P}(A)$ is a finite set with 2^n members.*** \square

15.2 The union of two sets

Definition 32 Let A, B be sets. The union of A and B is the set $A \cup B$ given by

$$A \cup B = \{x : x \in A \vee x \in B\}.$$

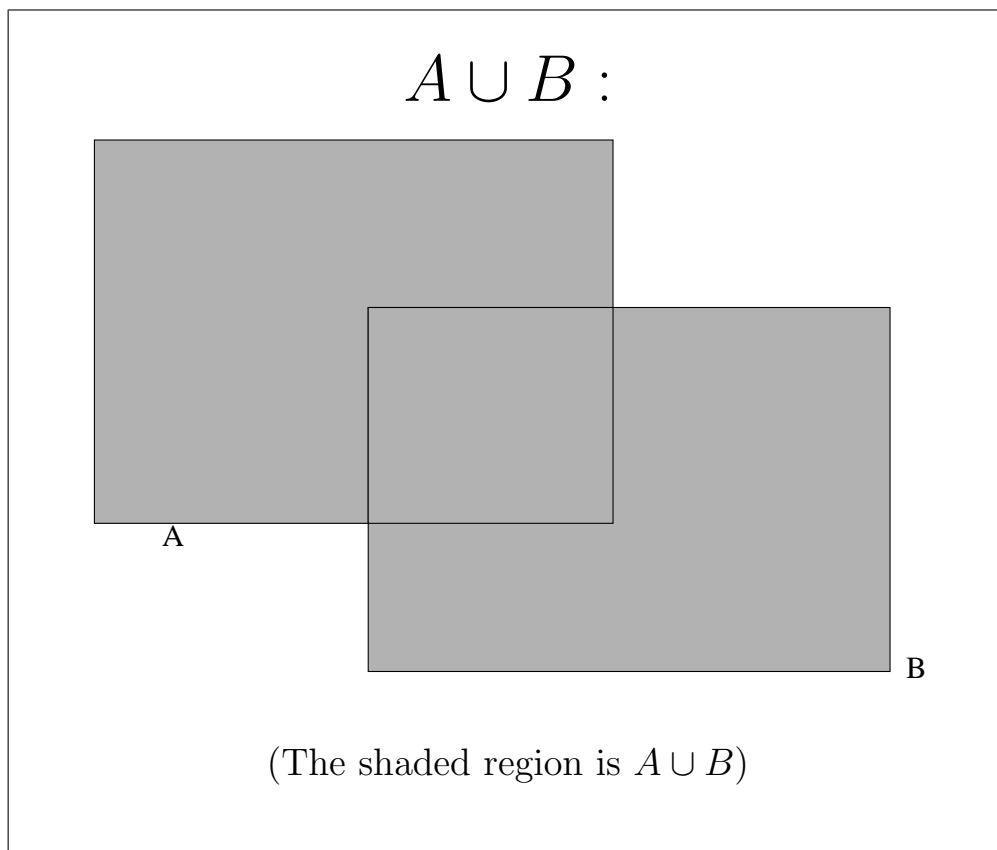
In other words, $A \cup B$ (read as “*A union B*”) is the set whose members are all the members of A as well as all the members of B .

The **membership criterion** for $A \cup B$ is “ $x \in A \vee x \in B$.” That is, for an object x to qualify as a member of $A \cup B$, it has to be shown that x is in A or that x is in B .

Example 54.

- If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ then $A \cup B = \{1, 2, 3, 4\}$.
- If $A = \{a, b, c\}$ and $B = \{d, e, f, g, h, i, j\}$ then $A \cup B = \{a, b, c, d, e, f, g, h, i, j\}$.
Notice that

1. A is a finite set with 3 members,
2. B is a finite set with 7 members,
3. A and B have no members in common (that is, using the terminology of the next section, $A \cap B = \emptyset$),
4. and $A \cup B$ has turned out to be a finite set with 10 members. **This is not a coincidence. We will prove later that: if A , B are finite sets, A has m members, B has n members, and $A \cap B = \emptyset$, then the union $A \cup B$ is a finite set with $m + n$ members.**
5. If $A = \{n \in \mathbb{Z} : n > 0\}$ and $B = \{n \in \mathbb{Z} : n < 0\}$ then $A \cup B = \{n \in \mathbb{Z} : n \neq 0\}$.
6. $\mathbb{N} \cup \{0\}$ is the set of all nonnegative integers, i.e., the set $\{n \in \mathbb{Z} : n \geq 0\}$.
7. If $A = \{x \in \mathbb{R} : 0 < x < 1\}$ and $B = \{x \in \mathbb{R} : 1 \leq x < 2\}$ then $A \cup B = \{x \in \mathbb{R} : 0 < x < 2\}$.
8. If $A = \{x \in \mathbb{R} : 0 < x < 1\}$ and $B = \{x \in \mathbb{R} : 1 < x < 2\}$ then $A \cup B = \{x \in \mathbb{R} : 0 < x < 2 \wedge x \neq 1\}$. \square



15.3 The intersection of two sets

Definition 33 Let A , B be sets. The intersection of A and B is the set $A \cap B$ given by

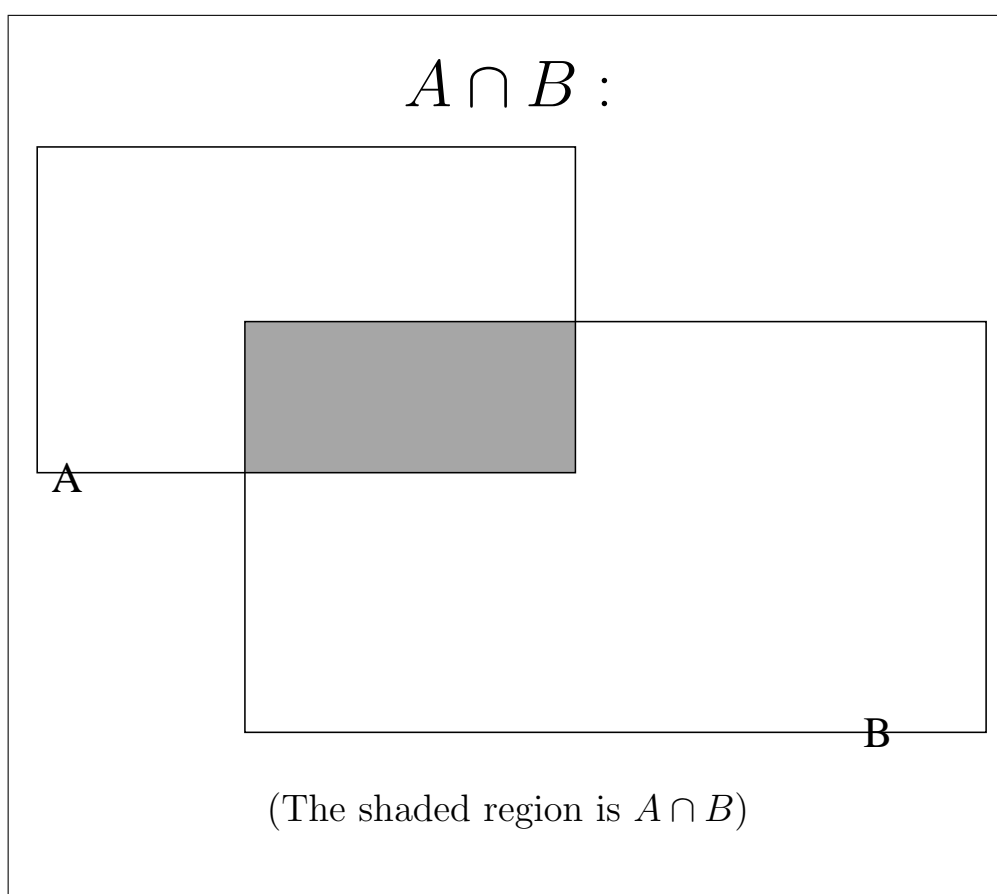
$$A \cap B = \{x : x \in A \wedge x \in B\}.$$

In other words, $A \cap B$ (read as “ A intersection B ”) is the set whose members are all the things that belong both to A and to B .

The **membership criterion** for $A \cap B$ is “ $x \in A \wedge x \in B$.” That is, for an object x to qualify as a member of $A \cap B$, it has to be shown that x is in A and that x is in B .

Example 55.

- If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ then $A \cap B = \{2, 3\}$.
- If $A = \{n \in \mathbb{Z} : n > 0\}$ and $B = \{n \in \mathbb{Z} : n < 0\}$ then $A \cap B = \emptyset$.
- If $A = \{x \in \mathbb{R} : 0 < x < 2\}$ and $B = \{x \in \mathbb{R} : 1 < x < 3\}$ then $A \cap B = \{x \in \mathbb{R} : 1 < x < 2\}$.



15.4 The difference of two sets

Definition 34 Let A, B be sets. The difference of A and B is the set $A - B$ given by

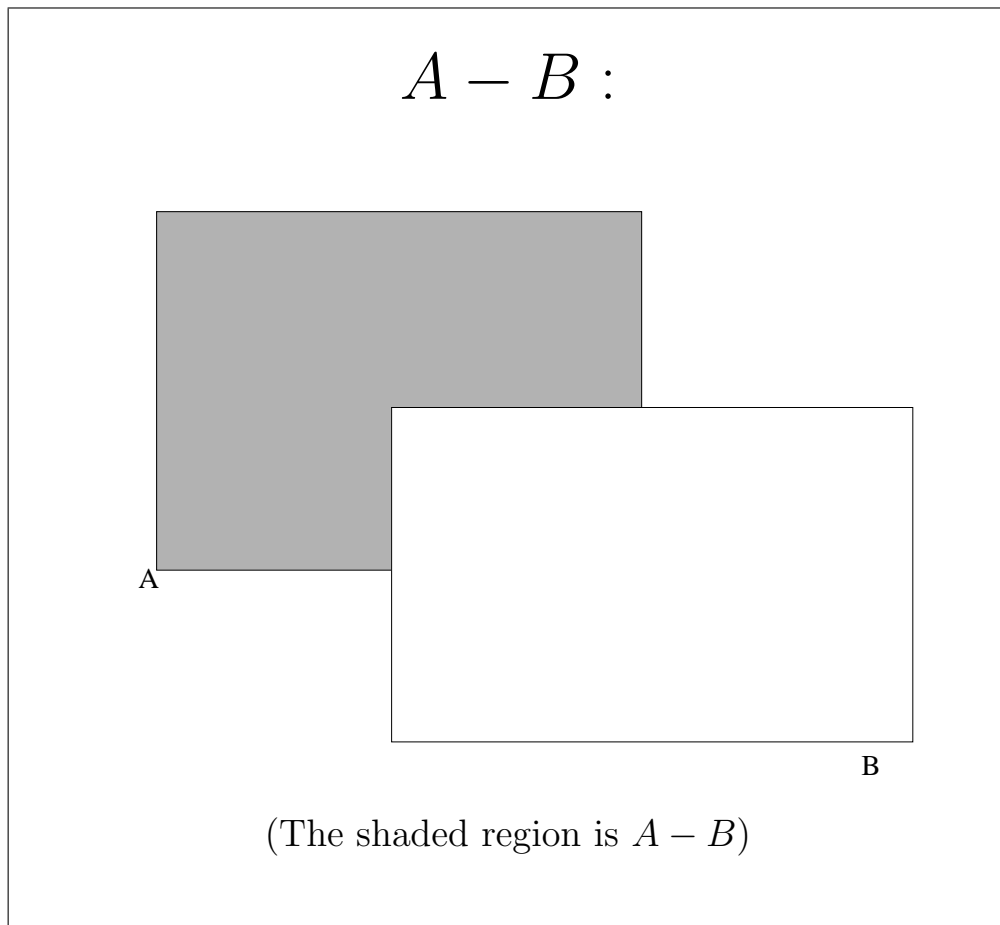
$$A - B = \{x : x \in A \wedge x \notin B\}.$$

In other words, $A - B$ (*read as “A minus B”*) *is the set whose members are all the things that belong to A but do not belong to B.*

The *membership criterion* for $A - B$ is “ $x \in A \wedge x \notin B$.” That is, for an object x to qualify as a member of $A - B$, it has to be shown that x is in A and that x is not in B .

Example 56.

- If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ then $A - B = \{1\}$.
- If $A = \mathbb{Z}$ and $B = \mathbb{N}$ then $A - B = \{n \in \mathbb{Z} : n \leq -1\}$.
- If $A = \{x \in \mathbb{R} : 0 < x < 2\}$ and $B = \{x \in \mathbb{R} : 1 < x < 3\}$ then $A - B = \{x \in \mathbb{R} : 0 < x \leq 1\}$.



15.5 Complements

As you may have noticed, the operations of union and intersection are closely related to the logical connectives \vee and \wedge :

$A \cup B$ is the set of those x such that $x \in A \vee x \in B$

$A \cap B$ is the set of those x such that $x \in A \wedge x \in B$

Given this, which is the set operation that corresponds to the negation symbol \sim ? Since \sim is a unary connective (i.e., it can be applied to one sentence S to produce the sentence $\sim S$), the corresponding operation, let us call it $\#$, should be a unary operation defined as follows:

$\#A$ is the set of those x such that $\sim x \in A$.

In other words, $\#A$ should be the set of all the things that are not members of A . This set $\#A$ could be called the “complement” of A , and would

be defined by $\#A = \{x : x \notin A\}$.

Now, the set $\#A$ would be truly huge. For example, if $A = \{1, 2, 3, 4\}$, then $\#A$ would consist of all the things other than the numbers 1, 2, 3, 4. So the members of $\#A$ would be the natural numbers other than 1, 2, 3, 4 (that is, 5, 6, 7 and so on), as well as the integers that are not natural numbers, all the real numbers other than 1, 2, 3, 4, plus all the other things that are not the numbers 1, 2, 3, 4, that is, all the cows, sheep, giraffes, people, rocks, tables, planets, stars, cells, viruses, molecules, atoms, electrons, protons, quarks, black holes, books, teeth, jackets, socks, cars, planes, forks, knives, and on and on and on.

Usually, when we are doing mathematics, we are studying a specific “universe” of mathematical objects. For example, when we do number theory we study the natural numbers or the integers, when we do Calculus we work with the real numbers, and when we do Multivariable Calculus we work with \mathbb{R}^2 , the set of pairs of real numbers (i.e., the “ xy plane”) or \mathbb{R}^3 (the set of triples (x, y, z) of real numbers, i.e., “3-dimensional space”). If, for example, our “world” is \mathbb{R} , then when we have a set A of real numbers, i.e., a subset A of \mathbb{R} , we would be interested in the set of real numbers that are not in A . And this set is the difference $\mathbb{R} - A$. So we give the following definition:

Definition 35. Suppose U is a set that we regard as the “universe”, in the sense that we are only interested in sets that are subsets of U . Then the complement of a set A such that $A \subseteq U$ is the set A^c given by

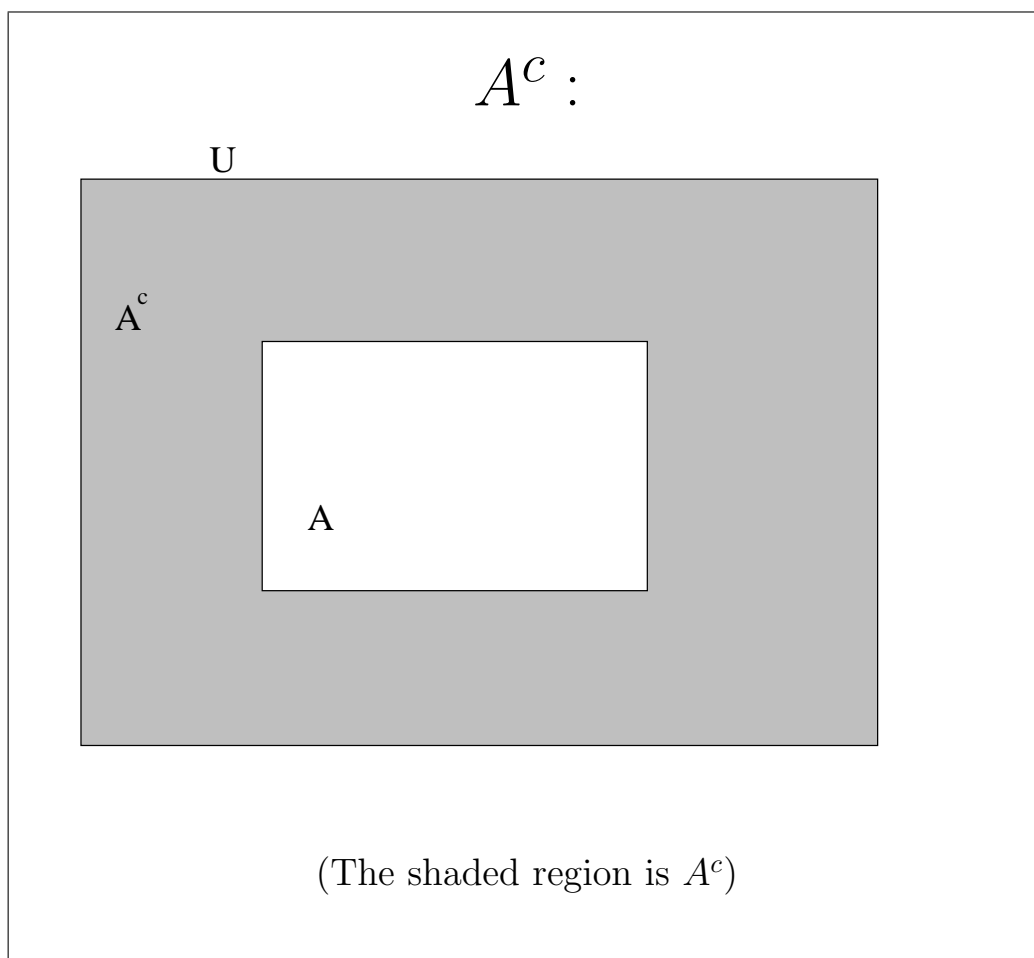
$$A^c = U - A, \quad (15.1)$$

that is,

$$A^c = \{x : x \in U \wedge x \notin A\}. \quad (15.2)$$

Remark 21. Strictly speaking, it is inappropriate to define a set as we did in Definition 35 and call it “ A^c ”. This set depends very much on who U is, so the right thing to do would be to call it the **complement of A relative to A** , and give it a name such as $A^{c,U}$, which shows that the set depends on U .

But, as long as we are working with a fixed “universe”, and it is clear who U is, it is O.K. to use a notation such as A^c . \square



15.6 The symmetric difference of two sets

Definition 36. Let A , B be sets. The symmetric difference of A and B is the set $A \Delta B$ given by

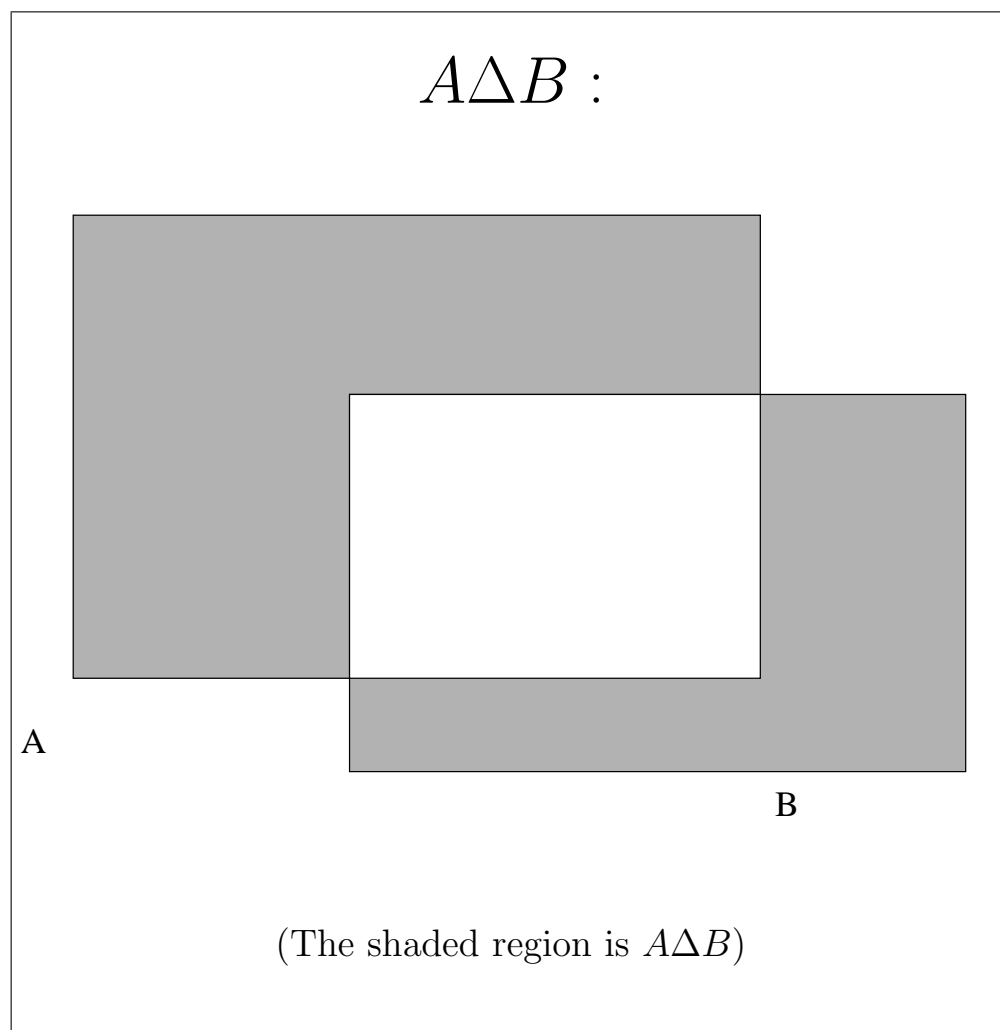
$$A \Delta B = \{x : (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)\}.$$

In other words, $A \Delta B$ (read as “the symmetric difference of A and B ”) is the set whose members are all the things that belong to A

but do not belong to B , or belong to B but do not belong to A .

That is, $A \Delta B$ *is the set of all things that belong to one of the sets A , B but do not belong to both.*

The **membership criterion** for the symmetric difference $A \Delta B$ is the sentence “ $(x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)$ ”. That is, for an object x to qualify as a member of $A - B$, it has to be shown that x is in A and that x is not in B , or that x is in B but not in A .



Example 57.

- If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ then $A \Delta B = \{1, 4\}$.

- If $A = \{x \in \mathbb{R} : |x| > 4\}$ and $B = \{x \in \mathbb{R} : |x| < 10\}$ then $A \Delta B = \{x \in \mathbb{R} : |x| \geq 10 \vee |x| \leq 4\}$.

15.7 The Cartesian product of two sets

15.7.1 Ordered pairs

If a, b are any two objects, we would like to have a set, called “the ordered pair of a and b ”, such that knowing this set would tell us who a is and who b is, so we would be able to say things such as “the first coordinate of (a, b) is a ” and “the second coordinate of (a, b) is b ”.

For example, suppose we are doing plane geometry, using the standard procedure of drawing an “ x axis” and a “ y axis”, and then representing each point P of the plane by a pair (a, b) of numbers, called the “coordinate pair” of P . Each point P then has, attached to it, a coordinate pair (a, b) of real numbers: the number a is the x **coordinate** (or “abscissa”) and the number b is the y **coordinate** (or “ordinate”) of P .

We would like the pair (a, b) to be a set, constructed somehow from a and b . And then the natural question is: which set is the pair (a, b) ?

The most naïve idea is to let the pair (a, b) be the unordered pair $\{a, b\}$, that is, the set whose members are a and b .

But this will not do. If we take (a, b) to be $\{a, b\}$, then it cannot happen, for example, that the x -coordinate of $(1, 2)$ is 1, and the x -coordinate of $(2, 1)$ is 2, because, if $(1, 2) = \{1, 2\}$ and $(2, 1) = \{2, 1\}$, then $(1, 2) = (2, 1)$, so, if

(*) the x -coordinate of $(2, 1)$ is 2,

then it would also be true that

(**) the x -coordinate of $(1, 2)$ is 2,

(because $(1, 2) = (2, 1)$), but on the other hand

(***) the x -coordinate of $(1, 2)$ is 1,

so we would get $1 = 2$, which is definitely not true.

The only solution is to define the ordered pair to be something other than the unordered pair $\{a, b\}$. And then the question is, **what set shall (a, b) be?**

There are many ways to answer this question, and it really makes no difference which one we use. So we shall choose one, but you must be warned that the specific way we choose is not important. What is important is that the following fact is true:

Theorem 53. *Let a, b, c, d be any objects. Then, if the pairs (a, b) and (c, d) are equal, that is, if $(a, b) = (c, d)$, it follows that $c = a$ and $d = b$.*

This is exactly the property that we need. For example, the pairs $(2, 1)$ and $(1, 2)$ are **not** equal. (Proof: Suppose $(2, 1) = (1, 2)$. Then Theorem 53 (with $a = 2$, $b = 1$, $c = 1$, and $d = 2$, would imply that $2 = 1$. But $2 \neq 1$. So $2 = 1 \wedge 2 \neq 1$, which is a contradiction. So $(2, 1) \neq (1, 2)$.)

Now we show how to define (a, b) in such a way that Theorem 53 is true.

Definition 37. Let a, b be any two objects. Then the ordered pair of a and b is the set (a, b) given by

$$(a, b) = \{ \{a\}, \{a, b\} \}. \quad (15.3)$$

Proof of Theorem 53. Suppose that $(a, b) = (c, d)$.

Let $p = (a, b)$, so p is also equal to (c, d) because we are assuming that $(a, b) = (c, d)$.

Since $p = \{ \{a\}, \{a, b\} \}$, the set p has either two members (if $b \neq a$) or one member (if $a = b$, in which case $\{a, b\} = \{a\}$, so $\{ \{a\}, \{a, b\} \} = \{ \{a\}, \{a\} \} = \{ \{a\} \}$).

But in either case, a is the only object that belongs to all the members of p . And, since p is also equal to (c, d) , it follows that c is the only object that belongs to all the members of p .

So $\boxed{c = a}$.

Next, let us prove that $d = b$.

We consider separately the two possible cases: $b = a$ and $b \neq a$.

Assume that $b = a$.

Then p has only one member, because, as explained before, $\{a, b\} = \{a\}$, so $p = \{ \{a\}, \{a, b\} \} = \{ \{a\} \}$.

But then (c, d) also has only one member, because $(c, d) = p$. And this implies that $d = c$.

So $d = c$ and $b = a$, and we already know that $c = a$.

Hence $\boxed{d = b}$.

Now assume that $b \neq a$.

Then the sets $\{a\}$ and $\{a, b\}$ are different, because $b \in \{a, b\}$ but $b \notin \{a\}$.

So p has two different members.

And b is the only object that belongs to one of the members of p but does not belong to both.

And, similarly, d is the only object that belongs to one of the members of p but does not belong to both.

So $\boxed{d = b}$.

We have proved that $d = b$ in both cases, when $b = a$ and when $b \neq a$.

So $\boxed{\boxed{d = b}}$.

So we have proved that $\boxed{\boxed{\boxed{c = a \wedge d = b}}}$. Q.E.D.

15.7.2 The Cartesian product of two sets

Definition 38. Let A, B be sets. The Cartesian product of A and B is the set $A \times B$ given by

$$A \times B = \left\{ u : (\exists a)(\exists b)(a \in A \wedge b \in B \wedge u = (a, b)) \right\}.$$

In other words, $A \times B$ (read as “ A times B ”) is the set of all objects u such that u is an ordered pair (a, b) , with $a \in A$ and $b \in B$.

Or, more succinctly and elegantly, $A \times B$ *is the set of all ordered pairs (a, b) for which $a \in A$ and $b \in B$.*

Example 58.

- Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$. Then

$$A \times B = \left\{ (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 3), (2, 4), (2, 5), \right. \\ \left. (3, 2), (3, 3), (3, 4), (3, 5) \right\}.$$

Notice that A is a finite set with 3 members, B is a finite set with 4 members, and $A \times B$ is a finite set with 12 members. ***This is not a coincidence. We will prove later that: if A, B are finite sets, A has m members, and B has n members, then $A \times B$ is a finite set and $A \times B$ has mn members.***

- Let $A = \mathbb{R}$, $B = \mathbb{R}$. Then $A \times B$ is $\mathbb{R} \times \mathbb{R}$, that is, the set of all ordered pairs (x, y) such that x and y are real numbers. ***This is the “ x - y plane” of plane Euclidean geometry. The members of $\mathbb{R} \times \mathbb{R}$ are the “points” of plane geometry.***

- Let

$$\begin{aligned} A &= \{x \in \mathbb{R} : 0 < x < 1\}, \\ B &= \{x \in \mathbb{R} : 1 < x < 3\}. \end{aligned}$$

Then A is the open interval $(0, 1)$ (***not to be confused with the ordered pair $(0, 1)$!***) and B is the open interval $(1, 3)$ (***not to be confused with the ordered pair $(1, 3)$!***). In this case, $A \times B$, that is, $(0, 1) \times (1, 3)$, is the set of all pairs (x, y) of real numbers such that $0 < x < 1$ and $1 < y < 3$. In other words, $(0, 1) \times (1, 3)$ ***is the rectangle R characterized by the inequalities***

$$0 < x < 1 \quad \text{and} \quad 1 < y < 3.$$

15.8 Important facts about the set operations

So far, we have defined:

- One very special set (the empty set),
- One binary predicate (i.e., relation), about sets, namely, the predicate “is a subset of”.
- Five binary operations on sets (union, intersection, difference, symmetric difference, and Cartesian product),
- One unary operation on sets (the power set).

By combining these nine things we can produce an enormous number of possible facts, some of which might be true, while others are not true. It would be pointless for me to give you a complete list and prove them all, because there are so many of them, and they are all so easy to prove (if true) or to disprove (if false).

And it would be pointless for you to memorize them all, because the list is so long. On the other hand, if you understand what you are doing, you ought to be able, in each case, to figure out if the statement is true or false, and how to prove it (if it is true) or disprove it (if it is false).

So what I suggest is this: ***read carefully the list of facts, and pick a few of them and prove them or disprove them. Keep in mind that any of these facts could show up as a question in the exams.***

And here is the list:

1. If A is a set, then $\emptyset \subseteq A$. (True)

2. If A is a set, then $\emptyset \in A$. (False)

3. If A is a set, then $A \cup \emptyset = A$. (True)

NOTE: If you think that \emptyset is like the number 0, and the operation " \cup " is like addition, then this statement is analogous to the statement that $x + 0 = x$ for every real number x .

4. If A is a set, then $A \cup \emptyset = \emptyset$. (False)

5. If A is a set, then $A \cap \emptyset = A$. (False)

6. If A is a set, then $A \cap \emptyset = \emptyset$. (True)

NOTE: If you think that \emptyset is like the number 0, and the operation " \cap " is like multiplication, then this statement is analogous to the statement that $x \cdot 0 = 0$ for every real number x .

7. If A is a set, then $A \subseteq A$. (True)

8. If A, B are sets, then $A = B$ if and only if $A \subseteq B \wedge B \subseteq A$. (True)

NOTE: This gives another way to prove that two sets are equal: to prove that $A = B$, you prove that $A \subseteq B$ and that $B \subseteq A$.

9. If A is a set, then $A \cup A = A$. (True)

10. If A is a set, then $A \cap A = A$. (True)

11. If A, B are sets, then $A \subseteq A \cup B$. (True)
12. If A, B are sets, then $A \subseteq A \cap B$. (False)
13. If A, B are sets, then $A \cup B \subseteq A$. (False)
14. If A, B are sets, then $A \cap B \subseteq A$. (True)
15. If A is a set, then $A \subseteq A$. (True)
NOTE: This says that the binary relation " \subseteq " is reflexive.
16. If A, B are sets, $A \subseteq B$, and $B \subseteq A$, then $A = B$. (True)
NOTE: This says that the binary relation " \subseteq " is antisymmetric.
17. If A, B, C are sets, $A \subseteq B$, and $B \subseteq C$, then $A \subseteq C$. (True)
NOTE: This says that the binary relation " \subseteq " is transitive.
18. If A, B, C are sets, $A \subseteq B$, $B \subseteq C$, and $C \subseteq A$, then $A = B = C$. (True)
19. If A, B are sets, then $A \subseteq B$ if and only if $A \cup B = B$. (True)
20. If A, B are sets, then $A \subseteq B$ if and only if $A \cup B = A$. (False)
21. If A, B are sets, then $A \subseteq B$ if and only if $A \cap B = A$. (True)
22. If A, B are sets, then $A \subseteq B$ if and only if $A \cap B = B$. (False)
23. If A, B are sets, then $A \cup B = B \cup A$. (True)
*NOTE: This is the **commutative law of the union operation**.*
24. If A, B are sets, then $A \cap B = B \cap A$. (True)
*NOTE: This is the **commutative law of the intersection operation**.*
25. If A, B, C are sets, then $A \cup (B \cup C) = (A \cup B) \cup C$. (True)
*NOTE: This is the **associative law of the union operation**.*
26. If A, B, C are sets, then $A \cap (B \cap C) = (A \cap B) \cap C$. (True)
*NOTE: This is the **associative law of the intersection operation**.*
27. If A, B, C are sets, and $A \subseteq B$, then $A \cup C \subseteq B \cup C$. (True)

28. If A, B, C are sets, and $A \subseteq B$, then $A \cap C \subseteq B \cap C$. (True)
29. If A, B, C are sets, then $(A \cup B) \cap C = A \cup (B \cap C)$. (False)
30. If A, B, C are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. (True)
*NOTE: This is the **distributive law of union with respect to intersection**.*
31. If A, B, C are sets, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. (True)
*NOTE: This is the **distributive law of intersection with respect to union**.*

*IMPORTANT NOTE: We have seen that union and intersection are in some ways like addition and multiplication: they obey commutative and associative laws. and also $A \cap \emptyset = \emptyset$ (which is analogous to $x \cdot 0 = 0$) and $A \cup \emptyset = A$ (which is analogous to $x + 0 = x$). But **the analogy should not be pushed too far:***

- *there is a distributive law of union with respect to intersection (i.e., $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$),*
- *and there is also a distributive law of intersection with respect to union (i.e., $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$),*
- *but this is totally unlike what happens for addition and multiplication, because*
- *there is a distributive law of multiplication with respect to addition (i.e., $x \cdot (y + z) = x \cdot y + x \cdot z$)*
- *but there is no distributive law of addition with respect to multiplication (i.e., it is not true that $x + (y \cdot z) = (x + y) \cdot (x + z)$ since, for example, if we take $x = 1, y = 2, z = 3$, then $x + (y \cdot z) = 7$ and $(x + y) \cdot (x + z) = 12$).*

32. If A, B are sets, then $(A - B) \cup B = A$. (False)
33. If A, B are sets, then $(A - B) \cup B \subseteq A$. (False)
34. If A, B are sets, then $A \subseteq (A - B) \cup B$. (True)
35. If A, B, C are sets, then $A \cup (B - C) = (A \cup B) - (A \cup C)$. (False)

36. If A, B, C are sets, then $A \cap (B - C) = (A \cap B) - (A \cap C)$. (False)

When we fix a “universe” U , then the complement of a subset A of U is defined to be the set $U - A$. The complement of A is denoted by “ A^c ”.

37. If A, U are sets, and $A \subseteq U$, then $(A^c)^c = A$. (True)

38. If A, U are sets, and $A \subseteq U$, then $A \cup A^c = U$. (True)

39. If A, U are sets, and $A \subseteq U$, then $A \cap A^c = \emptyset$. (True)

40. If A, B, U are sets, $A \subseteq U$, and $B \subseteq U$, then

$$(A \cup B)^c = A^c \cap B^c. \quad (15.4)$$

(This is true.)

41. If A, B, U are sets, $A \subseteq U$, and $B \subseteq U$, then

$$(A \cap B)^c = A^c \cup B^c. \quad (15.5)$$

(This is true.)

*NOTE: Equations (15.4) and (15.5) are the famous **De Morgan laws**. They say that*

- *the complement of the union of two sets is the intersection of the complements of the sets,*

and

- *the complement of the intersection of two sets is the union of the complements of the sets.*

I strongly recommend that you read the article on “De Morgan laws” in Wikipedia.

42. If A, B, U are sets, $A \subseteq U$, and $B \subseteq U$, then $A - B = A \cap B^c$. (True)

43. If A, B are sets, then $A - B = B - A$. (False)

44. If A, B, C are sets, then $A - (B - C) = (A - B) - C$. (False)

45. If A, B are sets, then $A \Delta B = (A \cup B) - (A \cap B)$. (True)

46. If A, B are sets, then $A \Delta B = B \Delta A$. (True)

47. If A, B, C are sets, then $A \Delta (B \Delta C) = (A \Delta B) \Delta C$. (True)

48. If A, B are sets, then $A \times B = B \times A$. (False)

49. If A, B, C, D are sets, then

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

(This is true.)

50. If A, B, C, D are sets, then

$$(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D).$$

(This is false.)

51. If A is a set, then $A \in \mathcal{P}(A)$. (True)

52. If A is a set, then $A \subseteq \mathcal{P}(A)$. (False)

53. If A is a set, then $\emptyset \in \mathcal{P}(A)$. (True)

54. If A is a set, then $\emptyset \subseteq \mathcal{P}(A)$. (True)

55. If A, B are sets, then $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. (True)

56. If A, B are sets, then $A = B$ if and only if $\mathcal{P}(A) = \mathcal{P}(B)$. (True)

57. If A, B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$. (False)

58. If A is a set, then $\emptyset \times A = \emptyset$ and $A \times \emptyset = \emptyset$. (True)

59. If A, B are sets, and $A \times B = B \times A$, then $A = B$. (False)

60. If A, B are nonempty sets, and $A \times B = B \times A$, then $A = B$. (True)

15.9 Some examples of proofs about sets

Let me give you the proofs of some of the results in the long list of the previous section.

15.9.1 Proof of one of the distributive laws

Theorem 54. *If A, B, C are sets, then*

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \quad (15.6)$$

Proof. To prove that the sets $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$ are equal, we prove that they have the same members, that is, we prove that

$$(\forall x) \left(x \in A \cup (B \cap C) \iff x \in (A \cup B) \cap (A \cup C) \right). \quad (15.7)$$

Sentence (15.7) is a universal sentence, of the form $(\forall x)P(x)$. So, in order to prove it, we let x be an arbitrary object and prove $P(x)$.

Let x be arbitrary.

We want to prove

$$x \in A \cup (B \cap C) \iff x \in (A \cup B) \cap (A \cup C). \quad (15.8)$$

- (1) The sentence “ $x \in A \cup (B \cap C)$ ” is equivalent to “ $x \in A \vee x \in B \cap C$ ”. (Reason: if X, Y are sets, then the criterion for membership in $X \cup Y$ is “ $x \in X \vee x \in Y$ ”.)
- (2) And “ $x \in B \cap C$ ” is equivalent to “ $x \in B \wedge x \in C$ ”. (Reason: if X, Y are sets, then the criterion for membership in $X \cap Y$ is “ $x \in X \wedge x \in Y$ ”.)
- (3) Hence “ $x \in A \cup (B \cap C)$ ” is equivalent to “ $x \in A \vee (x \in B \wedge x \in C)$ ”.
- (4) Also, “ $x \in (A \cup B) \cap (A \cup C)$ ” is equivalent to “ $x \in A \cup B \wedge x \in A \cup C$ ”.

And

- “ $x \in A \cup B$ ” is equivalent to “ $x \in A \vee x \in B$ ”.
 - “ $x \in A \cup C$ ” is equivalent to “ $x \in A \vee x \in C$ ”.
- (5) So “ $x \in (A \cup B) \cap (A \cup C)$ ” is equivalent to “ $(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$ ”.

It follows from (3) and (6) that “ $x \in A \cup (B \cap C) \iff x \in (A \cup B) \cap (A \cup C)$ ”, the sentence that we have to prove, is equivalent to

$$x \in A \vee (x \in B \wedge x \in C) \iff \left((x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \right). \quad (15.9)$$

The sentence (15.9) is of the form

$$P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R), \quad (15.10)$$

where P stands for “ $x \in A$ ”, Q stands for “ $x \in B$ ”, and R stands for “ $x \in C$ ”.

We now prove that (15.10) is true.

Sentence (15.10) is a biconditional, of the form $\mathcal{L} \iff \mathcal{M}$. And a biconditional $\mathcal{L} \iff \mathcal{M}$ is true if and only if \mathcal{L} and \mathcal{M} have the same truth value, i.e., are both true or both false. So we are going to prove that \mathcal{M} is true if \mathcal{L} is true and \mathcal{M} is false if \mathcal{L} is false.

Suppose that $P \vee (Q \wedge R)$ is true.

Then either P is true or $Q \wedge R$ is true.

Suppose P is true.

Then both $P \vee Q$ and $P \vee R$ are true.

So $(P \vee Q) \wedge (P \vee R)$ is true.

Now suppose that $Q \wedge R$ is true.

Then both Q and R are true.

So $P \vee Q$ and $P \vee R$ are true.

And then $(P \vee Q) \wedge (P \vee R)$ is true.

So $(P \vee Q) \wedge (P \vee R)$ is true in both cases, and then $(P \vee Q) \wedge (P \vee R)$ is true.

This proves that $(P \vee Q) \wedge (P \vee R)$ is true if $P \vee (Q \wedge R)$ is true.

Now suppose that $P \vee (Q \wedge R)$ is false.

Then both P and $Q \wedge R$ are false.

Since $Q \wedge R$ is false, either Q is false or R is false.

Suppose Q is false.

Since P is false, $P \vee Q$ is false, because both P and Q are false.

Hence the conjunction $(P \vee Q) \wedge (P \vee R)$ is false.

Now suppose R is false.

Since P is false, $P \vee R$ is false, because both P and R are false.

Hence the conjunction $(P \vee Q) \wedge (P \vee R)$ is false.

So $(P \vee Q) \wedge (P \vee R)$ is false in both cases, and then $(P \vee Q) \wedge (P \vee R)$ is false.

This proves that $(P \vee Q) \wedge (P \vee R)$ is false if $P \vee (Q \wedge R)$ is false.

So we have proved that (15.9) is true, and this completes our proof, **Q.E.D.**

Problem 41. *Prove the other distributive law: If A, B, C are sets, then*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \quad (15.11)$$

15.9.2 Proofs of the De Morgan laws

As we explained before, the De Morgan laws are the following two statements.

Theorem 55. *Let U be a set, and let A, B be subsets of U . Then*

$$(A \cup B)^c = A^c \cap B^c,$$

and

Theorem 56. *Let U be a set, and let A, B be subsets of U . Then*

$$\begin{aligned} (A \cup B)^c &= A^c \cap B^c, \\ (A \cap B)^c &= A^c \cup B^c. \end{aligned}$$

I will give you a proof from first principles¹ of the first theorem, and then I will give you a short proof of the other using the first one, and ask you to give a proof from first principles of the second theorem.

¹A *proof from first principles* is a proof in which you do not use any intermediate results proved before. For example, after we proved that $2 + 2 = 4$ from first principles we proved that $2 \times 2 = 4$ using the result that $2 + 2 = 4$. That was **not** a proof from first principles. In a proof from first principles, you would just have used the basic facts and the definitions, and no theorem proved before.

Proof. We want to prove that

$$\text{DeMorgan}(\forall x \in U) \left(x \in (A \cup B)^c \iff x \in A^c \cap B^c \right). \quad (15.12)$$

The sentence we want to prove is a universal sentence, of the form $(\forall x)P(x)$. So in order to prove it we let x be an arbitrary object and prove $P(x)$.

Let x be an arbitrary member of U .

We want to prove that

$$x \in (A \cup B)^c \iff x \in A^c \cap B^c. \quad (15.13)$$

But, for $x \in U$, " $x \in (A \cup B)^c$ " is equivalent to " $x \notin A \cup B$ ", i.e., to " $\sim x \in A \cup B$ ".

And " $x \in A \cup B$ " is equivalent to " $x \in A \vee x \in B$ ".

So " $x \notin A \cup B$ " is equivalent to " $\sim (x \in A \vee x \in B)$ ".

Therefore " $x \in (A \cup B)^c$ " is equivalent to " $\sim (x \in A \vee x \in B)$ ".

On the other hand, " $x \in A^c \cap B^c$ " is equivalent to " $x \in A^c \wedge x \in B^c$ ".

And the sentences " $x \in A^c$ ", " $x \in B^c$ " are equivalent to " $\sim x \in A$ " and " $\sim x \in B$ ".

So " $x \in A^c \cap B^c$ " is equivalent to " $(\sim x \in A) \wedge (\sim x \in B)$ ".

Hence (15.13) is equivalent to

$$\left(\sim (x \in A \vee x \in B) \right) \iff \left((\sim x \in A) \wedge (\sim x \in B) \right). \quad (15.14)$$

If we use P to stand for " $x \in A$ ", and Q to stand for " $x \in B$ ", then (15.14) is the sentence

$$\left(\sim (P \vee Q) \right) \iff \left((\sim P) \wedge (\sim Q) \right). \quad (15.15)$$

The biconditional sentence (15.15) is of the form $\mathcal{L} \iff \mathcal{M}$. And a biconditional $\mathcal{L} \iff \mathcal{M}$ is true if and only if \mathcal{L} and \mathcal{M} have the same truth value, i.e., are both true or both false. So we are going to prove that \mathcal{M} is true if \mathcal{L} is true and \mathcal{M} is false if \mathcal{L} is false.

Proof that if $\sim (P \vee Q)$ is true then $(\sim P) \wedge (\sim Q)$ is true.

Suppose that $\boxed{\sim (P \vee Q) \text{ is true}}$.

Then $P \vee Q$ is false.

So both P and Q are false.

Hence $\sim P$ and $\sim Q$ are true.

So the conjunction $\boxed{(\sim P) \wedge (\sim Q) \text{ is true}}$.

Proof that if $\sim (P \vee Q)$ is false then $(\sim P) \wedge (\sim Q)$ is false.

Suppose that $\boxed{\sim (P \vee Q) \text{ is false}}$.

Then $P \vee Q$ is true.

So either P is true or Q is true.

Suppose that P is true.

Then $\sim P$ is false.

So the conjunction $(\sim P) \wedge (\sim Q)$ is false.

Now suppose that Q is true.

Then $\sim Q$ is false.

So the conjunction $(\sim P) \wedge (\sim Q)$ is false.

We have shown that $(\sim P) \wedge (\sim Q)$ is false in both cases, when P is true and when Q is true.

Hence $\boxed{(\sim P) \wedge (\sim Q) \text{ is false}}$.

So we have proved (15.13) for an arbitrary member x of U , and we can go to

$$(\forall x \in U) \left(x \in (A \cup B)^c \iff x \in A^c \cap B^c \right). \quad (15.16)$$

And (15.16) says that the sets $(A \cup B)^c$ and $(A \cap B)^c$ have the same members, so the sets are equal, that is

$$(A \cup B)^c = A^c \cap B^c. \quad (15.17)$$

This is exactly what we wanted to prove.

Q.E.D.

Now let us give a simple proof of Theorem 56 using Theorem 55.

Proof. We want to prove that $(A \cap B)^c = A^c \cup B^c$.

Theorem 55 says that, if X, Y are any subsets of U , then

$$(X \cup Y)^c = X^c \cap Y^c. \quad (15.18)$$

Apply this with $X = A^c$ and $B = Y^c$. We get

$$(A^c \cup B^c)^c = (A^c)^c \cap (B^c)^c. \quad (15.19)$$

But $(A^c)^c = A$, and $(B^c)^c = B$. So

$$(A^c \cup B^c)^c = A \cap B. \quad (15.20)$$

Now take the complement of both sides. We get

$$\left((A^c \cup B^c)^c\right)^c = (A \cap B)^c. \quad (15.21)$$

But $(X^c)^c = X$ for every subset X of U . Therefore

$$\left((A^c \cup B^c)^c\right)^c = A^c \cup B^c \quad (15.22)$$

Combining (15.21) and (15.22), we get

$$A^c \cup B^c = (A \cap B)^c, \quad (15.23)$$

which is the formula we were trying to prove.

Q.E.D.

Problem 42. *Write a proof from first principles of Theorem 56. I strongly recommend that you use the same style as in the proof of Theorem 55. The proof of Theorem 55 is really very simple, and almost mechanical. It looks long because it was written on purpose to show you a proof written in a very precise, very detailed way, displaying the use of the rules of logic. Usually one does not write proofs like that, but I would like you to do it at least once, to show that you can do it.* \square

15.9.3 A proof involving the symmetric difference

Let us prove Fact 45 from our list. Recall that the *symmetric difference* of two sets A, B is the set $A\Delta B$ given by

$$A\Delta B = (A - B) \cup (B - A).$$

In the proof, we are going to use the following facts, that are valid for arbitrary subsets X, Y, Z of a set U :

- $X - Y = X \cap Y^c$,
- $X \cup X^c = U$ and $X \cap X^c = \emptyset$,
- $X \cap U = X$ and $X \cap \emptyset = \emptyset$.
- $X \cup U = U$ and $X \cup \emptyset = X$.
- The commutative laws

$$\begin{aligned} X \cup Y &= Y \cup X, \\ X \cap Y &= Y \cap X, \end{aligned}$$

- The associative laws

$$\begin{aligned} X \cup (Y \cup Z) &= (X \cup Y) \cup Z, \\ X \cap (Y \cap Z) &= (X \cap Y) \cap Z, \end{aligned}$$

- The distributive laws

$$\begin{aligned} X \cup (Y \cap Z) &= (X \cup Y) \cap (X \cup Z), \\ X \cap (Y \cup Z) &= (X \cap Y) \cup (X \cap Z), \end{aligned}$$

- The De Morgan laws

$$\begin{aligned} X^c \cup Y^c &= (X \cap Y)^c, \\ X^c \cap Y^c &= (X \cup Y)^c. \end{aligned}$$

Theorem 57. *If A, B are sets, then $A\Delta B = (A \cup B) - (A \cap B)$.*

Proof. Choose as universe any set U such that $A \subseteq U$ and $B \subseteq U$. (For example, we could choose U to be $A \cup B$.)

Then

$$A \Delta B = (A - B) \cup (B - A) \quad (15.24)$$

$$= (A \cap B^c) \cup (B \cap A^c) \quad (15.25)$$

$$= \left((A \cap B^c) \cup B \right) \cap \left((A \cap B^c) \cup A^c \right) \quad (15.26)$$

$$= \left(B \cup (A \cap B^c) \right) \cap \left(A^c \cup (A \cap B^c) \right) \quad (15.27)$$

$$= \left((B \cup A) \cap (B \cup B^c) \right) \cap \left((A^c \cup A) \cap (A^c \cup B^c) \right) \quad (15.28)$$

$$= \left((B \cup A) \cap U \right) \cap \left(U \cap (A^c \cup B^c) \right) \quad (15.29)$$

$$= (B \cup A) \cap (A^c \cup B^c) \quad (15.30)$$

$$= (A \cup B) \cap (A^c \cup B^c) \quad (15.31)$$

$$= (A \cup B) \cap (A \cap B)^c. \quad (15.32)$$

So $A \Delta B = (A \cup B) - (A \cap B)$.

Q.E.D.

Problem 43. Write the justifications of each of the nine steps (15.24), (15.25), (15.26), (15.27), (15.28), (15.29), (15.30), (15.31), (15.32) of the proof of Theorem 57. \square

Problem 44. *Prove or disprove* each of the following distributive laws

1. The distributive law of intersection with respect to symmetric difference. If A, B, C are sets, then

$$A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C). \quad (15.33)$$

2. The distributive law of union with respect to symmetric difference. If A, B, C are sets, then

$$A \cup (B \Delta C) = (A \cup B) \Delta (A \cup C). \quad (15.34)$$

16 Definitions: how you should write them and how you should not write them

16.1 An example of a correctly written definition

Suppose you don't know what a prime number is. And suppose you are asked whether the numbers 1, 2, 6, 7, 10, 12, are "prime". Then you will probably not be able to answer the question, because you don't know what a "prime number" is. So you would answer with a question: *what is a prime number*", or *what does it mean for a number to be prime*?

To answer such a question, you need to know the **definition** of "prime number".

And here is the definition:

DEFINITION OF "PRIME NUMBER"

Let n be a natural number. We say that n is prime if $n \neq 1$ and the only natural numbers that are factors of n are 1 and n .

And here is another, equally correct, definition of "perfect number":

DEFINITION OF "PRIME NUMBER", VERSION II

A natural number n is prime if $n \neq 1$ and every natural number that is a factor of n is either equal to 1 or to n .

And here is a third, also completely correct, definition of "perfect number":

DEFINITION OF “PRIME NUMBER”,
VERSION III

A natural number n is prime if $n \neq 1$ and
 $(\forall q \in \mathbb{N}) \left(q|n \implies (q = 1 \vee q = n) \right)$.

And, finally, here is a fourth completely correct definition of “perfect number”:

DEFINITION OF “PRIME NUMBER”,
VERSION IV:

An integer n is a prime number if $n > 1$ and
 $(\forall q \in \mathbb{N}) \left(q|n \implies (q = 1 \vee q = n) \right)$.

16.2 How not to write a definition

Let us look now at some bad ways of writing the definition “prime number”.

The examples I am going to give you are representative of things students often write in exams. ***You should read these examples carefully, and then read the explanation of why these definitions are bad, so that you will learn not to write that way.***

Some of the definitions below are truly horrendous (and would get zero points on a scale from 0 to 10), while others are not 100% wrong but are not entirely correct either, and may get 5 points on a 0-10 scale, or maybe in some cases even 6 or 7. But ***you should understand why those definitions are bad, so you can learn how to write definitions correctly and get 10 points out of 10.***

Bad Definition 1. Prime number is when you cannot divide by any number other than by the number itself. □

Bad Definition 2. A prime number is a number that cannot be divided by any number other than 1 and itself. □

Bad Definition 3. A prime number is a natural number that cannot be divided by any number other than 1 and itself. \square

Bad Definition 4. A prime number is a natural number such that the only factors of the number are 1 and the number itself. \square

Bad Definition 5. A prime number is a natural number such that the only factors of n are 1 and n . \square

Bad Definition 6. A prime number is a natural number such that the only natural numbers that are factors of n are 1 and n . \square

Bad Definition 7. A prime number is a natural number such that $n > 1$ and the only natural numbers that are factors of n are 1 and n . \square

Bad Definition 8. A prime number is a natural number n such that $n > 1$ and the only natural numbers that are factors of n are 1 and n . \square

16.2.1 Analysis of bad definitions

Let us analyze our eight “bad definitions” and explain why they are bad.

The main question that we will ask, and the question that you should always ask, is: *using this definition, can I tell correctly if an object is what the definition says it is supposed to be?* (In this case, *can I tell correctly if an object is a prime number or not?*)

Notice that this question really amounts to two questions:

- (I) *Can I tell?*, that is, *does the definition tell me precisely what to do in order to find out if the answer is “yes” or “no”?*
- (II) Can I tell *correctly*?, that is, *when I do what the definition tells me to do, do I get the right answers?*

Question (I) is the *precision and clarity* question: does the definition tell me clearly and precisely what I am supposed to do in order to find out the answer?

Question (II) is the *correctness* question: If I do what the definition tells me to do, do I get the right answer?

These two questions are different. For example, if I were to define “prime number” as follows:

Bad Definition 9. A prime number is a natural number that is divisible by 2. □

Then this definition is completely clear and precise. It tells me that in order to find out if a number is prime, I have to see if it is divisible by 2. The problem with this definition is that it does not satisfy the *correctness* condition: if I apply the definition, say, to the number 6, I find that 6 is divisible by 2, so according to this definition 6 is prime, which is not true.

To assess a definition, you should always ask these two questions: is the definition clear and precise, so that when I want to apply it I know exactly what to do? And is it correct, in the sense that it gives me the right answers?

And, in order to answer the correctness question, you should *test* your definition by applying it to several examples and seeing whether it gives the right answer.

The simplest and most convincing way to establish that a definition is wrong is to give an example of something for which the definition gives the wrong answer. This is what we did when we discussed the

You should always ask these two questions, *especially about definitions you have written yourself*. And if what you wrote does not meet the two requirements of (1) precision and clarity and (2) correctness, then your definition is not acceptable and you must work on it until you get it right.

Now let us look at the eight bad definitions in our list.

1. Bad Definition 1 says: *Prime number is when you cannot divide by any number other than by the number itself.*

This is truly atrocious. Let us see why.

- First of all, when you say “prime number is”, you are suggesting that “prime number” is a condition of the world, such as “chaos”, or “peace”. You can say something like “peace is when people are not fighting”, or “chaos is when there is utter confusion”. Even these sentences are very bad English, but you can more or less figure out what they mean. (For example, when you see that people are fighting, you would say that “there is no peace here”, and when people stop fighting, you would say “now there is peace”.) Much better ways to say these things would be: “Peace

is the absence of war or other hostilities”, or “Peace is a state of affairs in which people are not fighting”.

- But in the case of “prime number”, the “prime number is when” construction does not make sense. Being a prime number is not some kind of state of affairs. It is a property of a specific kind of object, namely, numbers. So one has to use much more precise language, and start the definition with “A prime number is”, or “A number is prime if”.

If a definition starts with “such and such is when...” you can be sure it is wrong:

- *“Prime number is when...” is wrong.*
- *“Divisible is when...” is wrong.*
- *“Even number is when...” is wrong.*
- *“Power set is when...” is wrong.*
- *“Subset is when...” is wrong.*
- *“Intersection is when...” is wrong.*

A correct definition of “prime number” should start in one of the following ways:

- “A prime number is a natural number n such that”
- “Let n be a natural number. Then n is a prime number if”
- “Let n be a natural number. We say that n is prime if”
- “A natural number n is prime if”

In other words: ***at the beginning of the definition you have to introduce the object or objects that you will be talking about.*** In this example, you do this by indicating that you will be talking about a natural number, not about a real number or a cow or a fish or a river. And you may give that natural number a name, such as n .

2. Bad Definitions 1 and 2 talk about “numbers”. We have already quoted Bad Definition 1, and Bad definition 2 says: *A prime number is a number that cannot be divided by any number other than 1 and itself.*

This definition does not pass the “can I tell?” test. It tells me that to be a prime number an object has to be a “number”.

But ***“number” is a vague concept***, because there are lots of different kinds of “numbers”, so when you say “number” you could mean “natural number” (that is, the kind of number that you are used to calling “whole number”), “integer”, “rational number”, “real number”, “complex number”, or lots of other kinds of numbers that exist.

Never say “number” unless it is clear what kind of “number” you are talking about.

If I want to follow Bad Definition 2, then, when I am given a thing and want to find out if that thing is a prime number, the first I thing I have to do is find out if it is a “number”. But I cannot do that because I don’t know what a “number” is. So the definition fails the “can I tell?” test.

In a correct, intelligible definition, when you talk about a ‘number’, you have to make it clear what you mean by “number”.

This can be made clear in at least three ways:

- You can just say what kind of number your number is supposed to be. (For example, you could say “let n be a natural number”, or “let n be an integer”, or “let n be a rational number”, or “let n be a real number”.)
- You can make it clear at the beginning of your text that the word “number” is always going to mean “integer”, or “real number”, or whatever. If you do so, then you don’t need to repeat that you mean “integer”, or “real number”, or whatever, every time you say “number”.
- You may want to talk about different kinds of numbers simultaneously. And, in order to do that, you may declare, at the beginning of your text, that, for example, “in this chapter, the letters m, n, p, q will always stand for natural numbers, and the letters x, y, z, u, v, w will stand for real numbers”.

3. Bad definitions 1, 2, and 3, talk about “dividing by numbers”, and tell me that a number is prime if it cannot be divided by certain numbers. But this is very confusing.

- Actually, ***any number can be divided by any number (except zero)***. For example, I can divide 7 by 5, getting as a result the number $\frac{7}{5}$.
- So the issue is not whether “we can divide”, because we can almost always do that, but ***what kind of result we get when we divide***.
- When Bad Definition 3 tells me that I should see if a number “can be divided by numbers other than 1 and the number itself”, then I could try to apply the definition, for example, to the number 3, and I would immediately see that 3 can be divided by lots of numbers other than 1 and 3: I can divide 3 by 2 (and the result is $\frac{3}{2}$), I can divide 3 by 7 (and the result is $\frac{3}{7}$), I can divide 3 by 29 (and the result is $\frac{3}{29}$), and so on.

4. Bad Definitions 4 and 5 are a little bit better. Rather than talk about “dividing”, they talk about “factors”, which is more precise. because we have a precise definition of “factor”.

But that is not good enough. According to the definition of “factor”, a factor of an integer a is an integer b such that there exists an integer k for which $a = bk$. So, when Bad Definition 5 says that

A prime number is a natural number such that the only factors of n are 1 and n .

then ***this definition fails the correctness test: according to this definition 2 is not prime***, because 2 has other factors in addition to 1 and 2. Indeed, -1 and -2 are factors of 2 as well, since $2 = (-1) \times (-2)$ and $2 = (-2) \times (-1)$.

5. Bad Definition 6 is much better. It says that

A prime number is a natural number such that the only natural numbers that are factors of n are 1 and n .

This is quite close, but ***this definition still fails the correctness test, because it gives us wrong answers***. Indeed, according to this definition 1 is prime. But this is wrong: 1 is not prime².

6. With Bad Definition 7 we enter, for the first time, the “partial credit” zone. This definition is essentially correct, but it is not well written. It says that

A prime number is a natural number such that $n > 1$ and the only natural numbers that are factors of n are 1 and n .

The problem with this is that the definition talks about “ n ” but does not tell us who this “ n ” is. ***In a mathematical text, when you refer to an object using a letter name, this name has to be introduced first.***

²Why is 1 not prime? For the same reason why Pluto is not a planet. Mathematicians have decided not to call 1 “prime”, exactly as astronomers have decided not to call Pluto a planet. But this decision was made for good reasons, that will be discussed later in this course.

7. Bad Definition 8 does this: the symbol “ n ” is properly introduced when we are told that

A prime number is a natural number n such that $n > 1$ and the only natural numbers that are factors of n are 1 and n .

8. So Bad Definition 8 is nearly perfect. What is missing? Only one thing: ***in a definition, the word or phrase being defined must be highlighted is some way, to indicate that we are defining that word or phrase.*** And when we write by hand the way we highlight is by underlining. So, for example, in a definition of “prime number” the words “prime number” have to be underlined. And if we do that we get a correct definition. *A prime number is a natural number n such that $n > 1$ and the only natural numbers that are factors of n are 1 and n .*

16.2.2 Always highlight the definiendum

When you write a definition, you are defining a particular word or phrase. That word or phrase is called the *definiendum*. (This just means “the thing being defined.”) ***The definiendum should always be highlighted.***

In books, the authors do this by using Italics, or Boldface. But when we write by hand, it is hard to do Italics or Boldface, so we use underlining.

Look, for example, at any definitions you want in our textbook. Just open the book at random, at any page, and look at the definitions on that page. And, for each definition, ask yourself “what is this definition the definition of?” And, invariably, you will see that the term or phrase being defined is in **boldface**. (This is not just a peculiarity of our textbook. It’s done in every Mathematics book.) In my lecture notes, I use underlining rather than boldface. And when you write your homework or your exams, or when I write on the blackboard, it’s hard to do italics or boldface, so I use underlining instead, and you should do the same.

16.3 The general formats for definitions

In a definition, the word, symbol or phrase whose meaning we are trying to define is called the definiendum.

16.3.1 Step 1: Find out if the definiendum is a term or a sentence, and what its arguments are

In order to know how to write a definition of something, we first have to figure out two things:

1. Whether the definiendum is a *term* or a *sentence*.
2. What the *arguments* of the definiendum are.

Recall that

- A *term* is a word or symbol or phrase that stands for a thing. Terms are essentially the same things that in your English or linguistics classes you may have called “noun phrases”.
- A *sentence* is a word or symbol or phrase that makes an assertion that can be true or false. Sentences are essentially the same things as “predicates”, or “statements”.
- Terms and sentences have *values*.
- The value of a term is the thing the term stands for. For example the term “New York City” is New York City.
- The value of a sentence is its truth value. For example, the sentence “New York City is the capital of New York State” has the truth value “false”, because it is not true, but the sentence “Albany is the capital of New York State” has the truth value “true”, because it is true.
- If a term or sentence contains variables, then the term or sentence only has a value, or truth value, if the variables that occur in it have been assigned values. For example,
 - the term “ $x + y$ ” contains two variables, x and y . If we assign values to these variables, by saying something like “let $x = 5$, $y = 3$ ”, then the term “ $x + y$ ” has the value 8.
 - the sentence “ $x + y = z$ ” contains three variables, x , y , and z . If we assign values to these variables, by saying something like “let $x = 5$, $y = 3$, $z = 4$ ”, then the sentence “ $x + y = z$ ” has the truth value “false”, because $5 + 3$ is not equal to 4.

16.4 Step 2: Introduce the arguments

You must start your definition by introducing the arguments.

For example:

- If you want to define “prime number”. then you will see, first of all, that the definiendum is a sentence, “something is a prime number”. And it has one argument, because we say things such as “ n is a prime number”. What you want to explain to the readers is how to tell what the truth value of the definiendum is for any given value or values of the arguments. That is, you want to tell the readers under what conditions they should call a number n “prime”, that is, when they should say “ n is prime”. So your definition must start by saying something like “Let n be an integer”, or “let n be a natural number”, or “let n be a real number”. (Eventually, n will turn out to be a natural number anyhow. So you could start your definition by requiring n to be a natural number. But you can also require n to be an integer, and let the second part of the definition force n to be a natural number, for example by putting the requirement that $n > 1$. And you could even start by requiring n to be a real number, and then say later: “we say that n is a prime number if it is a natural number such that ...”.)
- “Divisible” is a two-argument sentence, because we say things such as “ m is divisible by n ”, and these things are true or false. So in the definition of “divisible” you want to tell the readers under what conditions they should say of two numbers m, n that “ m is divisible by n ”. And you must start by introducing the two numbers m and n , by saying something like “Let m, n be integers”.
- “Union” is a two-argument term, because we talk about “the union of two sets A, B ”, and that union is a thing, namely, a set. So in the definition of “union” you want to tell the readers what the set $A \cup B$ is, if we are given two sets A, B . So you must start by introducing the two sets A and B , by saying something like “Let A, B be sets”.
- “Subset” is a two-argument sentence, because we say things such as “ A is a subset of B ”, and this sentence is true or false. So in the definition of “subset” you want to tell the readers under what conditions they should say that “ $A \subseteq B$ ” is true, if we are given two sets A, B . And you

must start by introducing the two sets A and B , by saying something like “Let A, B be sets”.

- “Power set” is a one-argument term, because we talk about “the power set of a set A ”, and that power set is a thing, namely, a set. So in the definition of “power set” you want to tell the readers who the set $\mathcal{P}(A)$ is, if we are given a set A . So you must start by introducing the set A , by saying something like “Let A be a set”.
- “Derivative” is more complicated, because there are two different concepts of derivative:
 - We talk about “the derivative of a function f at a point a .” This is a two-argument term: the derivative of f at a is a real number. So your definition of “derivative of a function at a point” must start by saying something like “Let f be a function and let a be a real number”.
 - We talk about “the derivative of a function f .” This is a one-argument term: the derivative of a function f is another function, usually called f' . So your definition of “derivative of a function” must start by saying something like “Let f be a function”.
- “married” is also complicated, like “derivative”. because there are two different concepts of “married”:
 - We talk about “two people begin married to each other.” This is a two-argument sentence: if x and y are people, then “ x and y are married to each other” can be true or false. So your definition of “ x and y are married to each other” must start by saying something like “Let x, y be two persons”.
 - We talk about one person being married, and say things like “ x is married.” This is a one-argument sentence. So your definition of “married” must start by saying something like “Let x be a person”.

16.5 Step 3: Tell the readers how to find the value of the definiendum

Now that you have introduced the arguments, you have to tell your readers how they can determine the value of the definiendum for those arguments. That value will be a thing if the definiendum is a term, and a truth value if the definiendum is a sentence.

For example:

- In the definition of “prime number”, after you have said, for example, “Let n be a natural number”, you have to tell the readers how to figure out the value of the definiendum, for n . In this case, the definiendum is the sentence “ n is prime”, so you have to tell the readers what has to happen that will make that sentence true. You can say, for example: “We say that n is a prime number if $n \neq 1$ and $(\forall m \in \mathbb{N})(m|n \implies (m = 1 \vee m = n))$ ”.
- In the definition of “divisible”. after you have said “Let m, n be integers”, you have to tell the readers how to figure out the value of the definiendum, for m and n . In this case, the definiendum is the sentence “ m is divisible by n ”, so you have to tell the readers what has to happen that will make them say that the sentence is true. You can say, for example: “We say that m is divisible by n if $(\exists k \in \mathbb{Z})m = nk$.”
- In the definition of “union”. after you have said “Let A, B be sets”, you have to tell the readers how to figure out the value of the definiendum, for A and B . In this case, the definiendum is the term “ $A \cup B$ ”, which is the name of a set. So you have to tell the readers who that set is, by saying, for example: “the union of A and B is the set $A \cup B$ given by $A \cup B = \{x : x \in A \vee x \in B\}$.”
- In the definition of “power set”. after you have said “Let A be a set”, you have to tell the readers how to figure out the value of the definiendum, for the set A . In this case, the definiendum is the term “ $\mathcal{P}(A)$ ”, which is the name of a set. So you have to tell the readers who that set is, by saying, for example: “the power set of A is the set $\mathcal{P}(A)$ given by $\mathcal{P}(A) = \{X : X \subseteq A\}$.”

Problem 45. *Analyze critically* (and, in particular, assign a grade on a scale from³ 0 to 10) each of the following definitions:

1. Definition of “union”: *The union of two sets is what you get when you combine the sets.*
2. Definition of “union”: *The union of two sets is all combined members of the sets.*
3. Definition of “union”: *Let A, B be sets. Then $A \cup B = \{x : x \in A \vee x \in B\}$.*
4. Definition of “divisible”: *A number n is divisible if it can be divided evenly into many parts.*
5. Definition of “divisible”: *Let m, n be integers. We say that m is divisible by n if $\frac{m}{n}$ is an integer*
6. Definition of “prime number”: *A prime number is a number that is not divisible by anything.*
7. Definition of “prime number”: *A prime number is a natural number n such that n has exactly two positive integer factors.*

³You are allowed to give negative grades like -300 for particularly atrocious definitions. And, since the authors of these definitions are just figments of my imagination, you don’t have to worry about the danger that you might hurt their feelings, and should feel free to be very harsh