MATHEMATICS 300 — FALL 2017

Introduction to Mathematical Reasoning

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INSTRUCTOR'S NOTES

PART IX

Contents

20 An introduction to logic

20.1 First-order predicate calculus

The language most mathematicians use to talk about mathematical objects (numbers of various kinds, sets, functions, lists, points, lines, planes, curves of various kinds, spaces where we can do geometry, graphs, and millions of other things) is a ***first-order predicate calculus***.

So let us explain what this means.

- The language is a "predicate calculus" because we can use it to express predicates.

So let us review what "predicates" are.

20.1.1 Predicates

Remember that

> A ***predicate*** is a sentence[a] involving one or more (or zero) variables, in such a way that the sentence has a definite truth value[b] for each choice of values of the variables.
>
> [a] "Sentence" means the same as "statement", or "assertion".
>
> [b] The ***truth value*** of a sentence is "true" if the sentence is true and "false" if the sentence is false.

For example:

- "Alice likes Mark" is a zero-variables predicate. It is either true or false.
- "x likes Mark" is a one-variable predicate. It is true or false depending on who x is. For example, suppose that Alice likes Mark but Andrew does not like Mark. Then "x likes Mark" is true when $x =$Alice but "x likes Mark" is false when $x =$Andrew.
- "x likes y" is a two-variables predicate. It is true or false depending on who x and y are. For example, supoose that Alice likes Mark, Andrew does not like Mark, Andrew likes Alice, and Mark does not like Andrew. Then "x likes y" is true when $x =$Alice and $y =$Mark, and when $x =$Andrew and $y =$Alice, but "x likes y" is false when $x =$Andrew and $y =$Mark.

- "x likes y more than x likes z" is a three-variables predicate.
- "$2+2=4$" and "$2+2=5$" are zero-variables predicates. They are either true or false. (And, of course, "$2+2=4$" is true and "$2+2=5$" is false.)
- "$x>0$" and "$2|n$" are one-variable predicates. They are true or false depending on who x (or n) is. For example, "$x>0$" is true $x=3$ but is false for $x=-5$. And "$2|n$" is true for $n=4$ but is false for $n=5$.
- "$x>y$" and "$m|n$" are two-variables (i.e., binary) predicates. They are true or false depending on who x and y (or m and n) are. For example, "the sentence $x>y$" is true for $x=5$ and $y=4$, but is false for $x=5$ and $y=6$. And "$m|n$" is true for $m=3$ and $y=6$, but is not true for $m=3$ and $y=7$.
- "$x+y=z$", "$x+y>z$", and "$n|m+q^2$" are three-variables predicates. The predicate "$x+y=z$" is,true for $x=2$, $y=3$ and $z=5$, but is false for $x=2$, $y=3$ and $z=4$. The predicate "$x+y>z$" is true for $x=2$, $y=3$ and $z=4$. but is false for $x=2$, $y=3$ and $z=5$. The predicate "$n|m+q^2$" is true for $n=5$, $m=9$, and $q=6$, but is false $n=5$, $m=7$, and $q=6$.
- "$x+2y^2-z>u$" and "$a=bq+r$ and $0\leq r<|b|$" are four-variables predicates. The predicate "$x+2y^2-z>u$" is true for $x=2$, $y=4$. $z=3$, $u=4$, but is false for $x=2$, $y=1$. $z=3$, $u=3$, The predicate "$a=bq+r$ and $0\leq r<|b|$" is true for $a=23$, $b=5$, $q=4$ and $r=3$, but is false for $a=23$, $b=5$, $q=4$ and $r=2$.

20.2 Free and bound variables, quantifiers, and the number of variables of a predicate

As was explained in the previous section, in a predicate such as "$x>y$", the variables x,y are ***free variables***, that is, variables that are free to be given any value we want. We can plug in values for x and y, and for each choice of values the resulting sentence has a definite truth value, that is, is true or false.

<table><tr><td>
You should think of a predicate as a processing device, with several "input channels". The input channels are the open variables. Each input channel is open, in the sense that the entrance to the channel is open so you can can put things in, or free, in the sense that we are free to put things in there. Once you have put in a value for, say, the variable x, then x is no longer open: it becomes closed, or bound.
Once you have put in values in all the input channels, the device processes these inputs, and produces a answer: true, or false.
</td></tr></table>

If, on the other hand, the predicate "$x > y$" appears in a text after a statement such as

$$\text{Let } x = 5,\ y = 3.$$

then the variables x and y are no longer free: they are ***bound variables***[1], because they are "attached" to particular values.

We now look at another, very important way to turn free variables into bound variables.

Let us consider, for example, the predicates

$$(\forall y \in \mathbb{R})x + 2y^2 - z > u \tag{20.1}$$

and

$$(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})(a = bq + r\,,\ \text{and}\ \ 0 \leq r < |b|)\,. \tag{20.2}$$

You may think that these are four-variables predicates, because each one of them contains four variables. (Predicate (20.1) contains the variables x, y, z and u. Predicate (20.2) contains the variables a, b, a and r.)

But this is not right:

$$\boxed{\text{\textbf{\textit{(20.1) is a three-variables predicate}}, and \textbf{\textit{(20.2) is two-variables predicate.}}.}}$$

Let me explain.

[1]Bound variables are also called ***closed variables***, because they are not open: the "input channel" through which we can input values for the variables is closed.

20.2.1 An example: a predicate with three free variables and one bound variable

We first look at the predicate

$$(\forall y \in \mathbb{R})x + 2y^2 - z > u\,. \tag{20.3}$$

- The predicate (20.3) is built from the predicate "$x + 2y^2 - z > u$" by ***quantifying*** it, i.e., putting a universal quantifier $(\forall y \in \mathbb{R})$ in front.
- The unquantified predicate "$x + 2y^2 - z > u$" contains the variables x, y, z, u. These are four open variables.
- So, if you are asked the "truth question"

  Is "$x + 2y^2 - z > u$" true or false?

  then you have to reply with a question of your own:

  Who are x, y, z and u?

- But in the quantified predicate (20.3) ***the variable y is quantified***.
- So, if you are asked the "truth question"

  Is "$(\forall y \in \mathbb{R})x + 2y^2 - z > u$" true or false?

  then you have to reply with the question:

  Who are x, z and u?

- In the predicate "$x + 2y^2 - z > u$", the four variables x, y, z and u are open variables, that is, "slots", or "input channels", where you can put in (or "plug in") values for each of the variables.
- When you fill in the four slots by plugging in values for the variables, you get a ***proposition***, i.e., a sentence that has a definite truth value.

A ***proposition*** is a sentence with no open variables

So a proposition is just true or false, whereas a predicate with open variables is true or false depending on the values of the variables.

Example:

1. The sentence "$m \geq n$" has two open variables. It is true if, for example, $m = 3$ and $n = 1$, and it is false if, for example, $m = 3$ and $n = 4$.

2. The sentence "$(\forall m \in \mathbb{N}) m \geq n$" is true if, for example, $n = 1$, and it is false if, for example, $n = 2$. So this sentence has one open variable, namely, n.

3. The sentences
$$(\exists n \in \mathbb{N})(\forall m \in \mathbb{N}) m \geq n$$
and
$$(\forall n \in \mathbb{N})(\forall m \in \mathbb{N}) m \geq n$$
do not have any open variables. So they are propositions. The first one is true. (Reason: Take $n = 1$. Then for arbitrary $m \in \mathbb{N}$ $m \geq 1$.) The second one is false. (Reason: Take $m = 1$, $n = 2$. Then it is not true that $m \geq n$.)

- So, for example, if you plug in the values $x = 2$, $y = 4$, $z = 3$, $u = 4$, into the sentence
$$x + 2y^2 - z > u$$

you get the proposition

$$19 > 4,,$$

which is true.

- But in the quantified predicate "$(\forall y \in \mathbb{R})x + 2y^2 - z > u$", there is no y-slot. The three variables x, z and u are open variables, that is, slots or input channels where you can put in values. But y is not an open variable.

- When you fill in the slots by plugging in values for the three open variables, you get a proposition.

- So, for example, if you plug in the values $x = 2$, $z = -3$, $u = 4$, into the sentence

  $$(\forall y \in \mathbb{R})x + 2y^2 - z > u$$

  then you get the sentence

  $$(\forall y \in \mathbb{R})2 + 2y^2 + 3 > 4$$

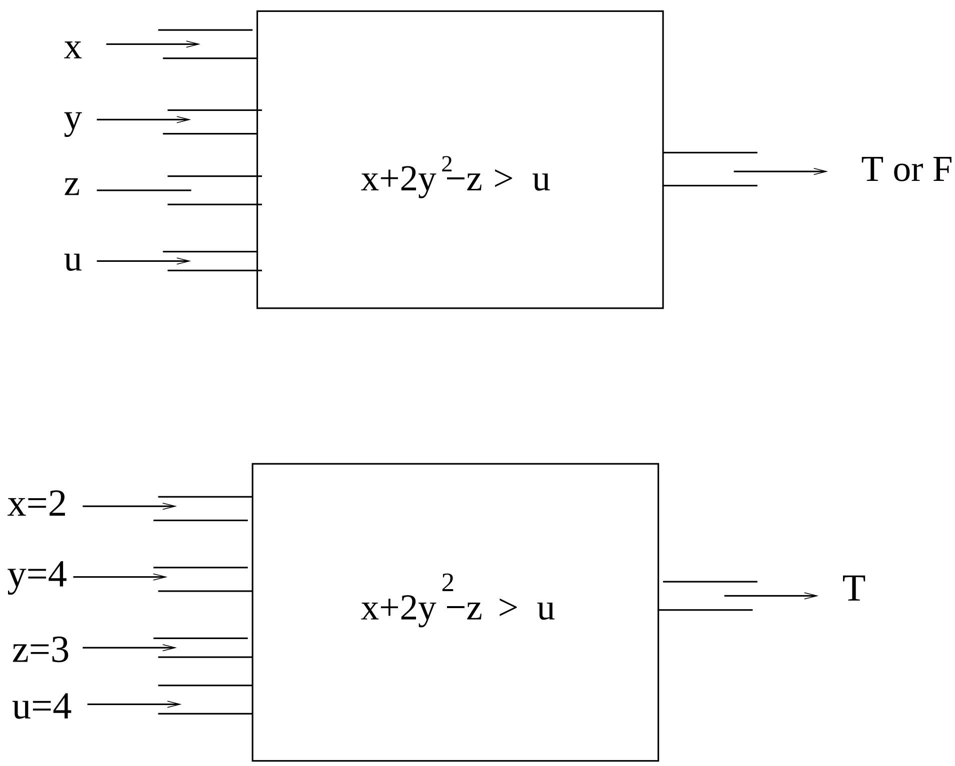
  which is equivalent to the sentence

  $$(\forall y \in \mathbb{R})2y^2 + 5 > 4\,.$$

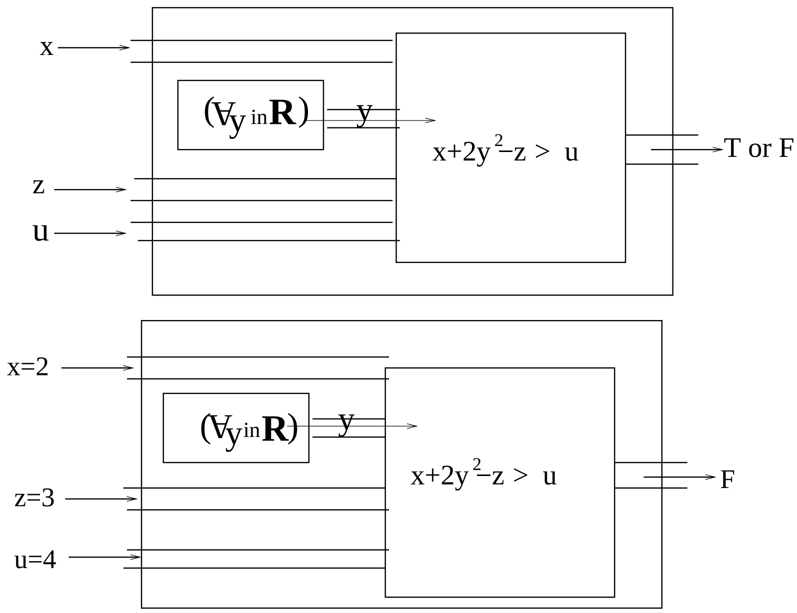
  And this sentence is true. (Proof: Let $y \in \mathbb{R}$ be arbitrary. Then $2y^2 \geq 0$. But $5 > 4$. So $2y^2 + 5 > 4$. Hence "$2y^2 + 5 > 0$" is true for arbitrary $y \in \mathbb{R}$. Therefore "$(\forall y \in \mathbb{R})2y^2 + 5 > 4$" is true.)

- The key point here is that ***the sentence "$(\forall y \in \mathbb{R})x + 2y^2 - z > u$" does not have a y-slot where you can plug in a value of y.*** That's because ***the sentence itself decides which value or values of y to plug in***. The quantifier $(\forall y \in \mathbb{R})$ says: "let y be an arbitrary real number". And then, with the values of x, z and u supplied by you, the truth value of the resulting sentence is determined. ***There is no need to ask "who is y?"***

Another way to see this is as follows: when you universally quantify a sentence by putting in front of it the universal quantifier "$(\forall y \in \mathbb{R})$", this adds to the sentence a "generator of y-values", that is, a new component that tells the sentence what value of y to use. More precisely, the universal quantifier "$(\forall y \in \mathbb{R})$" says "Let y be an arbitrary real number". And this ***closes*** the y-input channel, so that it is no longer possible to plug a y-value into the sentence from outside.



The sentence $\boxed{x + 2y^2 - z > u}$ is a processing device that has four input channels:the x-channel, the y-channel, the z-channel, and the u-channel. When values for the four variables are inputted into the sentence, the sentence produces a truth value. The four variables x, y, z, u are ***open***, or ***free***. They are open, because the input channels are open so that values of the variables can be put into them. They are free, because the variables are not tied to any particular value.



The quantified sentence $\boxed{(\forall y \in \mathbb{R})x + 2y^2 - z > u}$ is a combination of two interconnected processing units: the original unquantified sentence "$x + 2y^2 - z > u$", and the quantifier "$(\forall y \in \mathbb{R})$". The quantifier generates a value for the quantified variable y (by saying "let y be an arbitrary real number") and, by doing so, it ***closes*** the y input channel, so that y is no longer free; we cannot choose a value for y and plug it in. The other three channels remain open. So in this sentence x, z and u are open variables. but y is ***closed***, or ***bound***.

The other three letter variables (x, z and u) remain open. So we can plug in values for them in order to obtain propositions that have a definite truth value.

Summarizing:

- Even though the predicate "$(\forall y \in \mathbb{R})x + 2y^2 - z > u$" appears to contain four letter variables, only three of these variables (x, z and u) are open. The other variable, y, is ***bound***, or ***closed.***

- This means that the predicate "$(\forall y \in \mathbb{R})x + 2y^2 - z > u$" is a ***three variables***, or ***three arguments***, predicate. Therefore:

  - For each choice of values for x, z and u, the predicate becomes a proposition, i.e. a sentence with a definite truth value.

- If we want to give a name to this predicate, then we can of course call it P, but if we want to indicate the names of the free variables, we should call it $P(x,z,u)$.
- But ***we must not call it*** $P(x,y,z,u)$, because if we give it such a name we would erroneously be suggesting that this predicate has a "y-channel" where we can input values for the variable y.

- For example, "$(\forall y \in \mathbb{R})x + 2y^2 - z > u$" is true for $x = 4$, $z = 2$, $u = 1$. (Proof: We want to prove that $(\forall y \in \mathbb{R})4 + 2y^2 - 2 > 1$, that is, that $(\forall y \in \mathbb{R})2 + 2y^2 > 1$. Let $y \in \mathbb{R}$ be arbitrary. Then $y^2 \geq 0$, so $2y^2 \geq 0$, so $2 + 2y^2 \geq 2$, and $2 > 1$, so $2 + 2y^2 > 1$. Since "$2+2y^2 > 1$" has been proved to be true for arbitrary real y, it follows that $(\forall y \in \mathbb{R})2 + 2y^2 > 1$. Q.E.D.)

- The predicate "$(\forall y \in \mathbb{R})x + 2y^2 - z > u$" is false for $x = 4$, $z = 2$, $u = 8$. (Proof: We want to prove that "$(\forall y \in \mathbb{R})4 + 2y^2 - 2 > 1$" is not true, i.e., that "$(\forall y \in \mathbb{R})2 + 2y^2 > 8$" is not true. Take $y = 0$. Then "$2 + 2y^2 > 8$" is not true, because "$2 + 0 > 8$" is not true. So "$(\forall y \in \mathbb{R})4 + 2y^2 - 2 > 1$" is not true.Q.E.D.)

- The "truth question", i.e., the extra question we need to ask is order to be able to tell if "$(\forall y \in \mathbb{R})x + 2y^2 - z > u$" is true or false, is the question: ***"who are x, z and u?"***

- ***in order to have enough information to determine if the sentence "$(\forall y \in \mathbb{R})x+2y^2-z > u$" is true or false, we do not have to ask "who is y?", because once you are given the values of x, z and u, the quantified sentence itself determines if it is true or false, because it is up to the sentence to decide if it's true for all y or not, and it's not up to you to choose a value for y.***

20.2.2 A second example: a predicate with two free variables and two bound variables

We now look at the predicate

$$(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})(a = bq + r \wedge 0 \leq r < |b|)\,. \tag{20.4}$$

As I said before, on page 328, ***(20.2) is a two-variables predicate.***.

- Predicate (20.4) contains the variables a, b, q and r. But q ***and*** r ***are quantified***. So, if you are asked the "truth question"

$$\text{Is}\qquad \text{"}(\exists q\in\mathbb{Z})(\exists r\in\mathbb{Z})(a=bq+r\wedge 0\leq r<|b|)\text{" true or false?}$$

  then you have to reply with a question of your own:

  Who are a and b?

  The variables a and b in (20.4) are "slots", or "input channels", where you can put in (or "plug in") a value for each of the variables, and then you get a proposition.

- So, for example, if you plug in the values $a = 23$, $b = 11$, into the sentence
$$(\exists q\in\mathbb{Z})(\exists r\in\mathbb{Z})(a=bq+r\wedge 0\leq r<|b|)$$
  then you get the sentence
$$(\exists q\in\mathbb{Z})(\exists r\in\mathbb{Z})(23=11q+r\wedge 0\leq r<11)\,.$$
  And this sentence is true. (Proof: To prove an existential statement we use rule $\exists_{use}$: we exhibit values of q and r for which the proposition "$23 = 11q + r \wedge 0 \leq r < 11$" is true. Take $q = 2$, $r = 1$. Then $23 = 11q + r$ and $0 \leq r < 11$. Hence "$23 = 11q + r \wedge 0 \leq r < 11$" is true for some $q, r \in \mathbb{Z}$. Therefore "$(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})(23 = 11q+r\wedge 0 \leq r < 11$" is true.)

- The key point here is that ***the sentence***
$$(\exists q\in\mathbb{Z})(\exists r\in\mathbb{Z})(23=11q+r\wedge 0\leq r<11)$$
  does not have a q***-slot or an*** r***-slot where you can plug in values for*** q ***and*** r***.*** That's because ***the sentence itself decides***

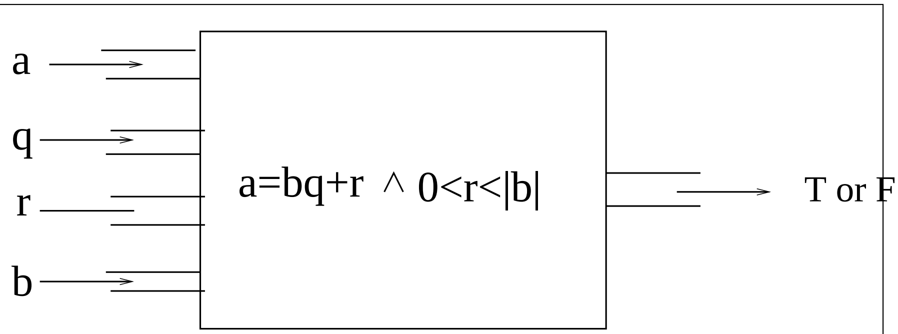
which value or values of* q *and* r *to plug in. The sentence itself[2] decides which values of q and r it has to look at, and then, with the values of a and b supplied by you, the truth value of the resulting sentence is determined.

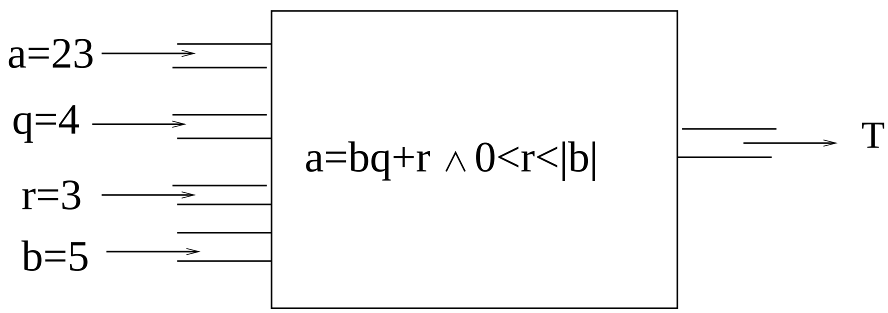
- Another way to see this is as follows: the sentence "$a = bq + r \wedge 0 \leq r < |b|$" has four input channels that are open, or free, so you can put into each channel a value of the corresponding variable.

  But when you existentially quantify the sentence twice by putting in front of it the two existential quantifiera "$(\exists q \in \mathbb{Z})$" and "$(\exists r \in \mathbb{Z})$", this adds to the sentence a "generator of q-values" and a "generator of r-values", that is, two new components that tell the sentence what values of q and r to look at. More precisely, the existential quantifiers "$(\exists q \in \mathbb{R})$" and "$(\exists r \in \mathbb{R})$" do the following:

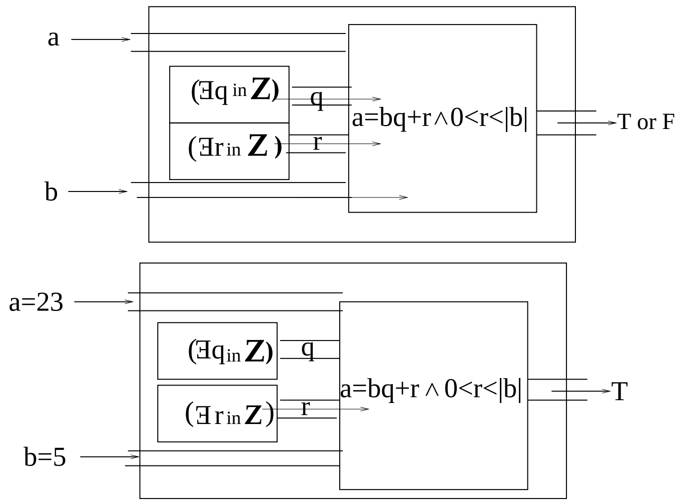
  - They look for a q-value and an r-value that make the sentence "$a = bq + r \wedge 0 \leq r < |b|$" true.
  - If they find such values, then they send to the sentence the message "yes, we have found values that make you true", and then the sentence produces the final verdict "yes, true".
  - If they do not find such values, then they send to the sentence the message "no, we have not found values that make you true", and then the sentence produces the final verdict "no, not true".

[2]Remember: you must think of a sentence as a processing device. The unquantified sentence "$a = bq+r \wedge 0 \leq r < |b|$" does the following: once it has been fed values for a, b, q and r, it finds out if "$a = bq+r \wedge 0 \leq r < |b|$" is true or not; if it is true is says "yes"; if it is false it says "no". The quantified sentence "$(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})(23 = 11q + r \wedge 0 \leq r < 11)$" does a much more complicated job: once it has been fed values for a and b, the sentence looks at all possible values of q and r, and sees whether it can find one choice of values of q and r for which "$23 = 11q + r \wedge 0 \leq r < 11$" is true; and then, if it find such values, it says "yes"; and if it cannot find any values of q and r for which "$23 = 11q + r \wedge 0 \leq r < 11$" is true, it says "no".





The sentence $\boxed{a = bq + r \wedge 0 \leq r < |b|}$ is a processing device that has four input channels:the a-channel, the q-channel, the r-channel, and the b-channel. When values for the four variables are inputted into the sentence, the sentence produces a truth value. The four variables a, q, r, b are ***open***, or ***free***. They are open, because the input channels are open so that values of the variables can be put into them. They are free, because the variables are not tied to any particular value.



The quantified sentence $\boxed{(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})(a = bq + r \wedge 0 \leq r < |b|)}$ is a combination of three interconnected processing units: the original unquantified sentence "$a = bq + r \wedge 0 \leq r < |b|)$", and the two quantifiers "$(\exists q \in \mathbb{Z}$", "$(\exists r \in \mathbb{Z}$". The quantifiers generate values for the quantified variablea q, r. (They look for values of q, r that will make "$a = bq + r \wedge 0 \leq r < |b|)$" true. If they find them, then they send one pair of such values to the main processing unit "$a = bq + r \wedge 0 \leq r < |b|)$", which then says "yes, true". If they do not find them, then they send some values to the main processing unit, but these values will not work, so the main processing unit wil say "no, not true".) By doing so, the quantifiers ***close*** the q and r input channels, so that q and f are no longer free; we cannot choose values for q and r and plug them in. The other two channels remain open. So in this sentence a and b are open variables. but q and r are ***closed***, or ***bound***.

The other two letter variables (a and b) remain open. So we can plug in values for them in order to obtain propositions that have a definite truth value.

Summarizing:

- Even though the predicate

$$(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})(a = bq + r \wedge 0 \leq r < |b|)$$

appears to contain four letter variables, only two of these variables (a and b) are open. The other variables, q and r, are ***bound***, or ***closed.***

- This means that the predicate

$$(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})(a = bq + r \wedge 0 \leq r < |b|)$$

is a ***two variables***, or ***two arguments***, predicate. Therefore:

  - For each choice of values for a and b, the predicate becomes a proposition, i.e. a sentence with a definite truth value. (And the Division Theorem tells us that the truth value is "true" for all choices of integers a and b such that $b \neq 0$, that is, that the proposition[3]

$$(\forall a \in \mathbb{Z})(\forall b \in \mathbb{Z})\Big(b \neq 0 \Longrightarrow \\ (\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})(a = bq + r \wedge 0 \leq r < |b|)\Big) \quad (20.5)$$

    is true.

  - Suppose we want to give a name to the two-variables predicate

$$(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})(a = bq + r \wedge 0 \leq r < |b|)\,.$$

    We can, of course, call it P. But if we want to indicate the names of the free variables, we should call it $P(a,b)$.

  - But ***we must not call it*** $P(a,b,q,r)$, because if we give it such a name we would erroneously be suggesting that this predicate has a "q-channel" and an "r-channel", where we can input values for the variables q, r.

- The "truth question", i.e., the extra question we need to ask is order to be able to tell if "$(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})(a = bq + r \wedge 0 \leq r < |b|)$" is true or false, is the question: ***"who are*** a ***and*** b***?"***

[3]Notice that (20.5) is a proposition, i.e., a predicate with no open variables at all (or, if you prefer, with zero open variables), because in (20.5) all four variables that occur are quantified, so a, b, q and r are closed variables. For the sentence (20.5), if you are asked "is this true", you do not need to ask any "truth question", because you do not need values of any variables to determine if the sentence is true.

- ***in order to have enough information to determine if the sentence "$(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})(a = bq + r \wedge 0 \leq r < |b|)$" is true or false, we do not have to ask "who are q and r?", because once you are given the values of a and b, the quantified sentence itself determines if it is true or false, because it is up to the sentence to decide if the required values of q and r exists or not, and it's not up to you to choose valuea for q and r.***

20.2.3 Another example, illustrating the fact that only open variables really matter

Some natural numbers are products of two prime numbers; for example, $4 = 2 \times 2$, $6 = 2 \times 3$, $35 = 5 \times 7$, and so on, Other natural numbers are not products of two prime numbers; for example, $18 = 2 \times 3 \times 3$, and the Fundamental Theorem of Arithmetic tells us that that there is no other way to write 18 as a product of primes, so in particular 18 is not the product of two primes.

So we can consider the predicate "n is a product of two prime numbers". And we can call this predicate $A(n)$. (We could just have called is "A", but we choose the name "$A(n)$" to emphasize the fact that this predicate has the open variable n.) Then, according to the conventions we made before about namis predicates, $A(6)$ is the proposition "6 is the product of two primes", and $A(7)$ is the proposition "7 is the product of two primes", so $A(6)$ is true, and $A(7)$ is false.

You can think of the predicate $A(n)$ as a "black box": you input a value of n, the predicate does some work, and produces an answer: "true" or "false". (For example, for $n = 6$ $A(n)$ is true, and for $n = 7$ $A(n)$ is false.)

But we can also look inside the box, and analyze in more detail how this predicate works. That is, we can observe that $A(n)$ says that

There exist prime numbers p, q such that $n = pq$.

So now our predicate has three variables, p, q, and n!

How come? Has the number of variables of $A(n)$ suddenly changed? Has $A(n)$ become a three-variables predicate? You may think so, because now $A(n)$ seems to have three variables: p, q and n.

But the answer is: ***No!*** $A(n)$ ***is still a one-variable predicate!*** The variables p and q are ***bound***, because they are quantified.

Precisely, $A(n)$ says, in formal language:

$$(\exists p \in \mathbb{N})(\exists q \in \mathbb{N})(p \text{ is prime} \wedge q \text{ is prime} \wedge n = pq)\,. \tag{20.6}$$

So, even though $A(n)$ appears to have three variables, namely, p, q and n, two of them are ***internal variables***[4], within the sentence (20.6). The sentence itself generates the values of p and q that it needs in order to answer its true-false question, and when the sentence ends p and q are free variables again. And, in particular, ***outside the sentence***

$$(\exists p \in \mathbb{N})(\exists q \in \mathbb{N})(p \text{ is prime} \wedge q \text{ is prime} \wedge n = pq)$$

the variables p ***and*** q ***have no value.***

Another way to see that p and q have no value, is to observe that $A(n)$ can equally well be written as

$$(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})(x \text{ is prime} \wedge y \text{ is prime} \wedge n = xy)\,, \tag{20.7}$$

or as

$$(\exists u \in \mathbb{N})(\exists v \in \mathbb{N})(u \text{ is prime} \wedge v \text{ is prime} \wedge n = uv)\,. \tag{20.8}$$

Sentences (20.6), (20.7), and (20.8) say exactly the same thing. The only difference is in the names of the variables that, inside the box, the sentence uses to process the inputs and produce an output.

From outside the box, we do not see these variables. ***That's why the letters*** p, q ***in (20.6), as well as the letters*** x, y ***in (20.7), and the letters*** u, v ***in (20.8), are internal variables, that have no value outside the sentence.***

And this is not the end of the story. "p is prime" is itself a complex predicate. In fact, "p is prine" stands for

$$p > 1 \wedge (\forall k \in \mathbb{N})\Big(k|p \Longrightarrow (k = 1 \vee k = p)\Big)\,. \tag{20.9}$$

[4]If you think of the sentence "$(\exists p \in \mathbb{N})(\exists q \in \mathbb{N})(p$ is prime)" as a processing unit, you will see that it has two existential quantifiers that generate values of p and q. But outside the processing unit all one sees is that certain values of n are fed in and certain 'true"s and "false"s come out. The variables p and q are part of the internal operation of the device.

This means that $A(n)$ can also be written as

$$(\exists p \in \mathbb{N})(\exists q \in \mathbb{N})\Bigg(\bigg(p > 1 \wedge (\forall k \in \mathbb{N})\Big(k|p \Longrightarrow (k = 1 \vee k = p)\Big)\bigg)$$
$$\wedge\bigg(q > 1 \wedge (\forall k \in \mathbb{N})\Big(k|q \Longrightarrow (k = 1 \vee k = q)\Big)\bigg) \wedge n = pq\Bigg). \tag{20.10}$$

Now one may think that $A(n)$ is a four-variables predicate, because it involves the variables n, p, q and k. But by now you know better: the new variable k is also bound, so the only open variable in (20.10)) is still n. That means that ***even if you write it in the form (20.10),*** $A(n)$ ***is still a one-variable predicate.***

Actually, the story doesn't end here either. "$k|p$" is also a complex preddcate with an internal structure of its own: is stands for "$(\exists j \in \mathbb{Z})p = kj$". So, if we substitute this for "$k|p$" in (20.10), we get an even more detalied version of $A(n)$, namely,

$$(\exists p \in \mathbb{N})(\exists q \in \mathbb{N})$$
$$\Bigg(\bigg(p > 1 \wedge (\forall k \in \mathbb{N})\Big((\exists j \in \mathbb{Z})p = kj \Longrightarrow (k = 1 \vee k = p)\Big)\bigg)$$
$$\wedge\bigg(q > 1 \wedge (\forall k \in \mathbb{N})\Big((\exists j \in \mathbb{Z})q = kj \Longrightarrow (k = 1 \vee k = q)\Big)\bigg)$$
$$\wedge n = pq\Bigg). \tag{20.11}$$

Now $A(n)$ apears to involve five variables: n, p, q, k and j. But this time you will have no problem figuring out that $A(n)$ ***is still a one-variable predicate, because the only open variable in (20.11) is still*** n***, and all the other variables are bound***.

Does this go on, or have we come to the end of our story? Actually, there is still on more step to go.

If you remember how we started our development of arithmetic (i.e., the theory of the integers and the natural numbers), you may[5] have noticed that in the basic facts about $\mathbb{Z}$ and $\mathbb{N}$ the symbol "$>$"never appears. We defined what "$>$" means later, and the definition was as follows:

[5] I am being polite here! What I really mean is you ***should*** have noticed.

$$\text{“}m > n\text{” means “}(\exists k \in \mathbb{N}) n = m + k\text{”}.$$

So in the sentence (20.11) we can eliminate all the occurrences of “>” using the definition, and we get

$$\begin{aligned}
&(\exists p \in \mathbb{N})(\exists q \in \mathbb{N}) \\
&\quad\Bigg(\bigg((\exists i \in \mathbb{N}) p = 1 + i \wedge (\forall k \in \mathbb{N})\Big((\exists j \in \mathbb{Z}) p = kj \Longrightarrow (k = 1 \vee k = p)\Big)\bigg) \\
&\quad\wedge\bigg((\exists i \in \mathbb{N}) q = 1 + i \wedge (\forall k \in \mathbb{N})\Big((\exists j \in \mathbb{Z}) q = kj \Longrightarrow (k = 1 \vee k = q)\Big)\bigg) \\
&\quad\wedge n = pq\Bigg). \qquad (20.12)
\end{aligned}$$

And now we have come to the end of the story. If you look at Formula (20.12), you will see that this way of expressing $A(n)$ involves six variables, namely, n, p, q, i, k and j. But this is still the same old $A(n)$, a one-variable predicate: only n is an open variable; the other five variables are closed.

Problem 41. *Draw a diagram of the sentence (20.12) as a processing unit, similar to the diagrams that appear on pages 333 and 338.*

Make sure that your diagram shows that there is only only one input channel. □

20.2.4 The primitive symbols of a theory

If you look carefully at sentence (20.12), you will notice that ***all the symbols***[6] ***are taken from the following list***:

[6] I am cheating a little bit, of course. Sentence (20.12) has parentheses of different sizes. But you should read all those different-sized parentheses as just parenteheses; the different sizes of parentheses are used just to make it easier for you to read he sentence.

The priimitive symbols of elementary integr arithmetic

The symbols are:

1. The letter variables (that is, $m, n, p, q, , r, s, u, v, x, y, a, b$ and so on.

2. The following 18 symbols:

Symbol	Name
(	left parenthesis
)	right parenthesis
$\in$	belongs to
$\mathbb{Z}$	the set of all integers
$\mathbb{N}$	the set of all natural numbers
$\forall$	the universal quantifier symbol
$\exists$	the existential quantifier symbol
$\sim$	negation
$\wedge$	conjunction
$\vee$	disjunction
$\Longrightarrow$	implication
$\Longleftrightarrow$	biconditional
$=$	equals
$+$	plus
$-$	minus
$\times$	times
0	zero
1	one

Every sentence of elementary integer arithmetic can be formulated as a sentence involving the symbols in the above list.

When we developed elementary arithmetic, we introduced many new symbols. But ***every new symbol was defined in terms of the pritive symbols***. So ***in every sentence involving new symbols it is possible to eliminate the new symbols and rewrite the sentence using primitive symbols only.***

20.2.5 Dummy variables

So far, we have seen that variables that appear in a sentence but are quantified are "internal variables", or "closed variables", or "bound variables". If you think of a sentence as a "processing unit", or "processing device", that takes in certain inputs and produces "true-false" outputs, then the closed (or bound, or internal) variables are variables that the sentence itself generates and uses to do its processing work. So the sentence does not need to be fed the values of these variables, and does not produce values of those variables that an outside obsevrer can see.

There is another way in which a variable appearing in a sentence can be a closed (or bound, or internal) variable. The sentence may contain a part that generates values of some variable in order to do a computation.

Consider. for example, the sentence

$$\sum_{k=1}^{n}(a + r^k) = b\,, \tag{20.13}$$

This sentence contains five letter variables, namely, a, r, b, k, and n.

Which ones of these five variables are open?

The best way to answer this question is by thinking of (20.13) as a processing device, opening it up to look into its internal structure, and figuring out what the device does.

Suppose you ask the device the truth question:

Is it true that $\sum_{k=1}^{n}(a + r^k) = b$?

Then the device will not know what to do, because in order to get started the device needs to be given the values of a, b, r, and n. (Maybe we should think of (20.13) as an inteligent device, that can ask questions. Then if you ask the truth question, the device will answer with a question: ***who are*** a***,*** b***,*** r ***and*** n***?***)

Suppose you do feed the device by inputting values for a, b,r and n. Then the device will do the following:

1. First, the CPU (central procssing unit) will report to the summation component Σ—that is, the component that computes the summation $\sum_{k=1}^{n}(a + r^k)$—the values of a, b, r and n that it has received from you.
2. Then Σ will perform the following calculation:

(a) First, it will write the list of all values of k, from 1 to n. (This is something it can do, because it knows who n is, since it has received this information from the CPU.)

(b) Then it will compute $a + r^k$ for each of the values of k in the list. (Again, Σ knows how to do this, because it knows who a and r are.)

(c) Then it will take all those values of $a + r^k$ that it has computed, and add them.

(d) Finally, it will report the result to the CPU. (Maybe, in order to facilitate communication between Σ and the CPU, they will introduce letter variables. For example, they may decide to call the result of the summation s, and then Σ will report the value of s to the CPU. But we need not concern ourselves with the variable s, because that's an internal variable used within the device for the various parts to communicate with each other.)

3. The CPU will then compare the result reported by the summation unit with b, and determine if they are equal.

4. If they are equal, the CPU will report to you the answer "true".

5. If they are not equal, the CPU will report to you the answer "false".

The main point of this is that k ***is an internal variable used by the sentence to perform its calculation. The values of*** k ***are generated by the sentence itself. So the sentence need not be given the value of*** k. And that's why

1. If asked the truth question, the sentence will ask "who are a, b, r and n""".

2. The sentence will not ask "who is k?", because ***the sentence itself generates the values of*** k ***it needs***.

3. k ***is not an open variable in (20.13)***

4. The open variables of (20.13) are a, b, r and n.

Let's just look at one more example. Let us analyze the following four sentences

$$(\forall n \in \mathbb{N})\Big((\exists m \in \mathbb{N})\sum_{k=1}^{m} k^3 = n \Longrightarrow (\exists p \in \mathbb{N})n = p^2\Big)\,, \tag{20.14}$$

$$(\forall n \in \mathbb{N})\Big((\exists m \in \mathbb{N})\sum_{k=1}^{m} k^3 = n \Longrightarrow (\exists p \in \mathbb{N})n = p^3\Big)\,, \tag{20.15}$$

$$(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})\Big(\sum_{k=1}^{m} k^3 = n \Longrightarrow (\exists p \in \mathbb{N})n = p^2\Big) \tag{20.16}$$

and

$$(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})\Big(\sum_{k=1}^{m} k^3 = n \Longrightarrow (\exists p \in \mathbb{N})n = p^3\Big) \tag{20.17}$$

Each of these sentences contains four variables, namely, n, m, k, and p.

And I am sure that this time you can see right away what is going on: ***all four variables are closed***. Three of them (n, m, and p) are quantified. and the variable k is also closed because the sentence itself generates the values of k that it needs to perfom its calculations.

So ***the sentences (20.14), (20.15), (20.16), and (20.17), are propositions***.

And then of course each of the sentences is true or false. Which leads me to a natural question, that I will ask you to answer.

Problem 42. *Which of the propositions (20.14), (20.15), (20.16), (20.17), are true, and which ones are false?*

NOTE: All these propositions are of the form $(\forall n \in \mathbb{N})P(n)$*, where* $P(n)$ *is a one-variable predicate having* n *as the open variable.*

If you want to prove that a sentence of this form is true, then you need a reasoned argument, starting with "Let n *be an arbitrary natural number." (You may also try a proof by induction, but in this case I would not recommend that.) If you want to prove that it is false, then you need a counterexample, i.e., an example of an* n *for which the one-variable sentence* $P(n)$ *is false.*

HINT: The answer to this problem is actually very easy. All you have to do is use the result of one of your earlier homework problems. (I can narrow this down a bit further: it's one of the problems in the third set of lecture notes.*) Using this, plus a little bit of logic (for example, truth values of implications), each of the four propositions should just require a couple of lines on your part.)* ☐

A variable such as the k in $\sum_{k=1}^{n} t(k)$ (where $t(k)$ is some expression containing k, such as k, or k^2, or r^k, or $a+r^k$), is called a "dummy variable".

Let us define this term precisely. (The definition I am about to give is taken from Wolfram MathWorld.)

Definition 57. A dummy variable is ***a variable that appears in a calculation only as a placeholder and which disappears completely in the final result.*** ☐

And ***every dummy variable is bounded***.

Example 46. Naturally, summations are not the only type of expressions where some of the variables are bound variables

Examples of dummy variables are:

1. the k in a summation such as $\sum_{k=1}^{n} t(k)$,
2. the k in a product such as $\prod_{k=1}^{n} t(k)$,
3. the k in the name of a list, such as $(p_k)_{k=1}^{n}$,
4. the x in the name $\{x : P(x)\}$ of a set,
5. the x in an integral such as $\int_a^b f(x)dx$.
6. the x in a limit such as $\lim_{x\to a} f(x)$. ☐

Example 47. Let us look at the sentence

$$(\exists a \in \mathbb{R})(\exists b \in \mathbb{R})\Big(\{u \in \mathbb{R} : a \leq u \leq b\} \neq \emptyset \wedge \int_a^b x^2 dx = c\Big)\,. \qquad (20.18)$$

This sentence contains the letter variables a, b, u, x, and c.

Of these five letters, four are bound variables:

1. the variables a and b are bound because they are quantified;

2. the variable u is bound because it is a dummy variable, used as part of the name $\{u \in \mathbb{R} : a \leq u \leq b\}$ of a set;

3. the variable x is bound because it is a dummy variable, used as a variable of integration.

It follows from this analysis that

1. ***Sentence (20.18) defines a one-variable predicate.***

2. ***The open variable in sentence (20.18) is*** c.

3. If you think of sentence (20.18) as a processing device, then this device will take values of c as inputs, and produce a true-false answer as output.

4. If you ask the "truth question" ***is (20.18) true?***, then the device (20.18) cannot answer because it does not know who c is. So the device will answer your question with another question: ***who is*** c***?***

5. But, in order to be able to answer the truth question, the device does not need to ask "who is a?", or "who is b?" or "who is u?", or "who is x?". The device itself will generate the values of a, b, u and x it needs, and these values will be part of the calculations that (20.18) performs, and will not be seen by the outside world.

20.2.6 How to tell if a variable is dummy

Here are two ways to see that a variable is dummy.

1. The variable is dummy if "it isn't really there", in the sense that we can eliminate it completely. For example,

   (a) The set $\{u \in \mathbb{R} : a \leq u \leq< b\}$ is an object very well known to all of us: it is none other than the closed interval $[a, b]$. So we can say the same thing as (20.18) by writing "$[a, b]$" instead of "$\{u \in \mathbb{R} : a \leq u \leq< b\}$". And we get

$$(\exists a \in \mathbb{R})(\exists b \in \mathbb{R})\Big([a,b] \neq \emptyset \wedge \int_a^b x^2 dx = c\Big)\,, \tag{20.19}$$

which says exactly the same thing as (20.18). but now there is no "u" anymore.

(b) The definite integral $\int_a^b x^2 dx$ is a number that is completely determined by a and b. We do not need to ask "who is x?" in order to determine this number. Actually, the integral can be computed, and the result is $\frac{1}{3}(b^3 - a^3)$. So we can say the same thing as (20.19) by writing "$\frac{1}{3}(b^3 - a^3)$" instead of "$\int_a^b x^2 dx$", and we get

$$(\exists a \in \mathbb{R})(\exists b \in \mathbb{R})\Big([a, b] \neq \emptyset \wedge \frac{1}{3}(b^3 - a^3) = c\Big), \qquad (20.20)$$

or, more nicely,

$$(\exists a \in \mathbb{R})(\exists b \in \mathbb{R})\Big([a, b] \neq \emptyset \wedge b^3 - a^3 = 3c\Big), \qquad (20.21)$$

which say exactly the same thing as (20.19). but now there is no "x" anymore.

2. A variable is dummy if it can be replaced by any other variable (except with variables that are already being used for something else) without changing the meaning of the sentence. For example,

(a) If instead of the expression "$\{u \in \mathbb{R} : a \leq u \leq b\}$" we use a different letter and write something like "$\{v \in \mathbb{R} : a \leq v \leq b\}$", or "$\{z \in \mathbb{R} : a \leq z \leq b\}$", or maybe "$\{\alpha \in \mathbb{R} : a \leq \alpha \leq b\}$", or "$\{\diamond \in \mathbb{R} : a \leq \diamond \leq b\}$", nothing changes. So, for example, we can rewrite (20.18) as

$$(\exists a \in \mathbb{R})(\exists b \in \mathbb{R})\Big(\{q \in \mathbb{R} : a \leq q \leq b\} \neq \emptyset \wedge \int_a^b x^2 dx = c\Big), \qquad (20.22)$$

which says exactly the same thing as (20.18). but now there is no u anymore.

(b) If we replace the definite integral $\int_a^b x^2 dx$ by the expression $\int_a^b h^2 dh$, or $\int_a^b \sigma^2 d\sigma$, or $\int_a^b m^2 dm$, nothing changes. So, for example, we can rewrite (20.22) as

$$(\exists a \in \mathbb{R})(\exists b \in \mathbb{R})\Big(\{q \in \mathbb{R} : a \leq q \leq b\} \neq \emptyset \wedge \int_a^b k^2 dk = c\Big), \qquad (20.23)$$

which says exactly the same thing as (20.18). but now there is no u and no x anymore.

Summarizing: Sentence (20.18) defines a ***one-variable predicate, with the open variable*** c. So we can call this predicate $P(c)$.

And then we may ask: can we tell what this predicate $P(c)$ is? Can we find a simpler expression for $P(c)$?

It turns out that, in this case, the answer is "yes, we can":

$\boxed{P(c) \text{ ***just says*** "}c \geq 0\text{"}}$.

Proof. We want to prove that $(\forall c \in \mathbb{R})(P(c) \Longleftrightarrow c \geq 0)$.

Let $c \in \mathbb{R}$ be arbitrary.

We want to prove that $P(c) \Longleftrightarrow c \geq 0$.

For that purpose, we will prove the implications $P(c) \Longrightarrow c \geq 0$ and $c \geq 0 \Longrightarrow P(c)$.

Proof that $P(c) \Longrightarrow c \geq 0$.

Assume $P(c)$.

This means that

$$(\exists a \in \mathbb{R})(\exists b \in \mathbb{R})\Big([a, b] \neq \emptyset \wedge b^3 - a^3 = 3c\Big).$$

Pick real numbers a, b such that $a, b] \neq \emptyset$ and $b^3 - a^3 = 3c$.

Since $a, b] \neq \emptyset$, it follows that $a \leq b$. (Reason: if $a > b$ then the set $[a, b]$, i.e., the set $\{u \in \mathbb{R} : a \leq u \leq b\}$, would be empty.)

Since $a \leq b$, we have $a^3 \leq b^3$.

So $b^3 - a^3 \geq 9$.

So $3c \geq 0$.

Hence $\boxed{c \geq 0}$.

Proof that $c \geq 0 \Longrightarrow P(c)$.

Assume that $c \geq 0$.

Let $a = 0$, $b = \sqrt[3]{3c}$.

Then $b \geq 0$.

So the closed interval $[a, b]$ (i.e., the interval $[0, b]$) is nonemtpy.

And $b^3 - a^3 = 3c$.

Hence $[a,b] \neq \emptyset \wedge b^3 - a^3 = 3c$.

So

$$(\exists a \in \mathbb{R})(\exists b \in \mathbb{R})\Big([a,b] \neq \emptyset \wedge b^3 - a^3 = 3c\Big).$$

That is, $\boxed{P(c) \text{ holds.}}$

Since we gave proved that $P(c) \Longrightarrow c \geq 0$ and that $c \geq 0 \Longrightarrow P(c)$, we can conclude that $P(c) \Longleftrightarrow c \geq 0$.

Since we have proved that $P(c) \Longleftrightarrow c \geq 0$ for arbitrary real c, we have proved that $(\forall c \in \mathbb{R})\Big(P(c) \Longleftrightarrow c \geq 0)$. **Q.E.D.**

20.2.7 Some important examples of common mistakes students make in induction proofs

Let us look at some common mistakes people make in proofs by induction.

Suppose, for example, that we are asked to prove by induction that

$$(\forall n \in \mathbb{N})\,\text{every tournament with } n \text{ players has a top player}. \qquad (20.24)$$

Some students often start their proof by writing this:

Let n be an arbitrary natural number.

Let $P(n)$ be the sentence

"every tournament with n players has a top player".

And then they go on from there.

This is wrong. Why? Because if you start by saying "Let n be an arbitrary natural number", then at this point you are declaring a value for n. Then from that moment on "n" is the name of a fixed number, so $P(n)$ is not a predicate with the open variable n, where n is a free variable, i.e., a variable to which we are free to assign any value we want.

Other students start their proof by writing this:

> Let $P(n)$ be the sentence "$(\forall n \in \mathbb{N})$ every tournament with n players has a top player".

And then they go on from there.

This is also wrong. Why? Because ***the sentence*** $P(n)$ ***that you use to do induction has to have*** n ***as an open variable,*** But the sentence "$(\forall n \in \mathbb{N})$ every tournament with n players has a top player" has no open variables at all: the variable n is bound (i.e., closed).

And there are students who start their proof by writing this:

> Let $P(n) =$ every tournament with n players has a top player.

And then they go on from there.

This is somewhat less horrendous a mistake than those of the two previous examples, but ***it is still something you should not write.*** Why? Because if you write "$P(n) =$ every tournament with n players has a top player" this looks like saying that the equality "$P(n) =$every". If the students want me to read what they wrote as saying that the sentence $P(n)$ is the same as the sentence "every tournament with n players has a top player", then ***the way to say that is to enclose the sentence "every tournament with*** n ***players has a top player" in quotation marks***, and write

> Lat $P(n)$ be the sentence "every tournament with n players has a top player"

without the equal sign.

Now let us look at another example.

Suppose that we are asked to prove by induction that

$$(\forall n \in \mathbb{N})(\forall m \in \mathbb{N})m!n!\Big|(m+n)!\,. \tag{20.25}$$

(This is a very important result. It says that the binomial coefficients $\binom{n+m}{m}$ are integers.) (Remember that $\binom{k}{j}$ is defined to be $\frac{k!}{j!(k-j)!}$. So, if we are given nonnegative integers m, n, and we let $k = m+n$, $j = m$, then $k - j = n$, and $\binom{n+m}{n} = \frac{(m+n)!}{m!n!}$.)

Students often start their proof by writing this:

Let $P(n)$ be the sentence "$m!n!\Big|(n+m)!$"

And then they go on from there.

This is wrong. Why? Because ***the sentence*** $P(n)$ ***that you use to do induction has to have*** n ***as an open variable and no other open variables***, but the sentence "$m!n!\Big|(n+m)!$" has two open variables.

Other students start their proof by writing this:

Let $P(n)$ be the sentence

$$(\forall n \in \mathbb{N})(\forall m \in \mathbb{N})m!n!\Big|(n+m)!\,. \tag{20.26}$$

And then they go on from there.

This is wrong. Why? Because ***the sentence*** $P(n)$ ***that you use to do induction has to have*** n ***as an open variable***, and this sentence has no open variables at all.

And there are students who start their proof by writing this:

Let $P(n) = m!n!\Big|(n+m)!$.

And then they go on from there.

This is not as bad a mistake as in the two previous examples, but ***it is still something you should not write.*** Why? Because if you write "$P(n) = m!n!\Big|(n+m)!$" then this can be read as asserting that $P(n)$ is equal to the number $m!$, or to the number $m!n!$. And this is nonsense, because $P(n)$ is a sentence and $m!$ and $m!n!$ are natural numbers, so there is no way on Earth that they could be equal to the sentence $P(n)$.

How am I supposed to know that the authors want me to see that $P(n)$ is the sentence "$m!n!\Big|(n+m)!$"? If the students want me to read what they wrote as saying that the sentence $P(n)$ is the same as the sentence "$m!n!\Big|(n+m)!$", then the way to say that is to enclose the sentence "$m!n!\Big|(n+m)!$" in quotation marks, and write

Lat $P(n)$ be the sentence "“$m!n!\Big|(n+m)!$”".

Problem 43. ***Write a correct, correctly written, proof*** *of (20.26). (HINT: This is basically a problem that you have already done before. It is nearly identical to Problem 49 in the fifth set of lecture notes.)* □

20.3 First-order predicate calculus

The language we use in mathematics is a ***predicate calculus*** because it enables us to predicates. And it is ***first-order*** because we can quantify variables, and write things such as "$(\forall x \in P)x$ likes Mark" (meaning, if P is the set of all people, "everybody likes Mark"), but we cannot quantify over predicates. That is,

- We cannot say things such as "'for every predicate $P(x)$ and every predicate $Q(x)$ if $(\forall x)P(x)$ is true and $(\forall x)Q(x))$ is true, then if $(\forall x)(P(x) \wedge Q(x))$ is true."
- We can say this for a particular pair of predicates $P(x)$, $Q(x)$ (for example, we can say "if everybody likes coffee and everybody likes milk then everybdoy likes coffee and milk", or we can say "if everybody studies and everybody reads books then everybdoy studies and reads books"), but we cannot say the same thing for arbitrary predicates $P(x)$, $Q(x)$.

It turns out that there are "second order" languages, in which you can say things like "'for every predicate $P(x)$ and every predicate $Q(x)$ if $(\forall x)P(x)$ is true and $(\forall x)Q(x))$ is true, then if $(\forall x)(P(x) \wedge Q(x))$ is true." But the language we are using here is a ***first-order language***, in which those things cannot be said.

20.4 Logical connectives

In firts-order predicate calculus, one or more sentences can be combined to form other sentences. The symbols used to combine sentences are called the ***logical connectives.*** And there are exactly seven of them

20.4.1 The seven logical connectives

And here they are, in all their glory:

The seven logical connectives

1. The ***negation symbol*** $\sim$ (meaning "no", "it's not the case that").
2. The ***conjunction symbol***, $\bigwedge$ (meaning "and").
3. The ***disjunction symbol***, $\bigvee$ (meaning "or").
4. The ***implication symbol***, $\Longrightarrow$ (meaning "implies", or "if ... then").
5. The ***biconditional symbol***, $\Longleftrightarrow$ (meaning "if and only if").
6. The ***existential quantifier symbol***, $\exists$ (meaning "there exists ... such that", or "it is possible to pick ... such that").
7. The ***universal quantifier symbol***, $\forall$ (meaning "for all",or "for avery", or "for an arbitrary").

20.4.2 How the seven logical connectives are used to form sentences

These seven symbols are used to form new sentences as follows:

1. The negation symbol $\sim$ is a ***one-argument connective***: it can be put in front of a sentence A to form the sentence $\sim A$ (meaning "no A", or "it's not the case that A"). For example: "$\sim 3|5$" means "3 does not divide 5".
2. The conjunction symbol $\wedge$ is a ***binary connective***, or ***two-argument connective***: it can be put between two sentences A, B to form the sentence $A \wedge B$, (meaning "A and B"). For example: "$(\sim 3|5) \wedge 3|6$" means "3 does not divide 5 and 3 divides 6".

3. The disjunction symbol $\wedge$ is a ***binary connective***, or ***two-argument connective***: it can be put between two sentences A, B to form the sentence $A \vee B$, (meaning "A or B"). For example: "$x > 0 \vee x < 0$" means "$x > 0$ or $x < 0$".

4. The implication symbol $\Longrightarrow$ is a ***binary connective***, or ***two-argument connective***: it can be put between two sentences A, B to form the sentence $A \Longrightarrow B$, (meaning "A implies B", or "if A then B"). For example: "$x \neq 0 \Longrightarrow x^2 > 0$" means "if $x > 0$ then $x^2 > 0$".

5. The biconditional symbol $\Longleftrightarrow$ is a ***two-argument connective***, that is ***binary connective***: it can be put between two sentences A, B to form the sentence $A \Longleftrightarrow B$, (meaning "A if and only if B"). For example: "$(2|n \wedge 3|n) \Longleftrightarrow 6|n$" means "2 divides n and 3 divides n if and only if 6 divides n".

6. The existential quantifier symbol $\exists$ has a more complicated grammar:

   (a) Using $\exists$ we can form ***existential quantifiers***.

   (b) There are two kinds of existential quantifiers:

      i. ***Unrestricted existential quantifiers*** are expressions
      $$\Big(\exists x\Big),$$
      that is: left parenthesis, $\exists$, variable, right parenthesis.

      ii. ***Restricted existential quantifiers*** are expressions
      $$\Big(\exists x \in S\Big),$$
      that is: left parenthesis, $\exists$, variable, $\in$, name of a set, right parenthesis.

   (c) Then we can take a sentence A (or $A(x)$) and put a restricted or unrestricted existential quantifier in front, forming the sentences $(\exists x)A$ ("there exists x such that A", or "it is possible to pick x such that A") and $(\exists x \in S)A$ ("there exists x belonging to S such that A", or "it is possible to pick x belonging to S such that A").

7. The universal quantifier symbol $\forall$ has a grammar similar to that of the existential quantifier symbol:

   (a) Using $\forall$ we can form ***universal quantifiers***.

(b) There are two kinds of universal quantifiers:

i. ***Unrestricted universal quantifiers*** are expressions
$$\big(\forall x\big),$$
that is: left parenthesis, $\forall$, variable, right parenthesis.

ii. ***Restricted universal quantifiers*** are expressions
$$\big(\forall x \in S\big),$$
that is: left parenthesis, $\forall$, variable, $\in$, name of a set, right parenthesis.

(c) Then we can take a sentence A (or $A(x)$) and put a restricted or unrestricted universal quantifier in front, forming the sentences $(\forall x)A$ ("for all x, A", or "A i strue for arbitrary x") and $(\forall x \in S)A$ ("for all x belonging to S, A", or "A is true for arbitrary x in S").

20.5 Conjunctions ("$\wedge$", i.e., "and")

The symbol
$$\bigwedge$$
is the ***conjunction symbol***, and means "and".

Hence,

- If P is the sentence

  Today is Friday

  and Q is the sentence

  Tomorrow is Saturday

  then "$P \wedge Q$" stands for the sentence

  Today is Friday and tomorrow is Saturday.

- A sentence of the form $P \wedge Q$ is a ***conjunction***.
- In a conjunction $P \wedge Q$, the sentences P, Q are the ***conjuncts***.

20.5.1 Proving a conjunction: a stupid but important rule

The rule for proving a conjunction (Rule $\wedge_{prove}$)

If P, Q are sentences, and you have proved P and you have proved Q, then you are allowed to go to $P \wedge Q$.

IMPORTANT REMARK. You may wonder "what is the point of such a rule?" But you cannot dispute that it is a reasonable rule! Of course, if you know that "today is Friday" and you also know that "tomorrow is Saturday", then you will have no doubt that "today is Friday and tomorrow is Saturday" is true. So you should have no problem accepting (and remembering) this rule. You may not understand why it is needed. So let me tell you why. Suppose it was a computer doing proofs, rather than a human being like you. Suppose the computer is told that today is Friday and then it is told that tomorrow is Saturday. How will the computer know that it can write "today is Friday and tomorrow is Saturday". It won't, unless you tell it. Computers do not "know" anything on their own. If you want the computer to "know" that once it knows that "today is Friday" and also that "tomorrow is Saturday", then it can write "today is Friday and tomorrow is Saturday", then you have to **tell** the computer. In other words, you have to input Rule $\wedge_{prove}$ into the computer. Proofs are mechanical manipulations of strings of symbols, and should therefore be doable by a computer. So Rule $\wedge_{prove}$ is needed.

And now let's go back to you, the human being. How do *you* know that, once you find out that "today is Friday" and also that "tomorrow is Saturday", then you can say (or write) "today is Friday and tomorrow is Saturday". **You know this because you know Rule** $\wedge_{prove}$. You know this rule so well, it is embedded so deeply in your mind, that you don't even realize that the rule is there. But **the rule is there!**

Here is another way to think about this. Suppose you didn't know any English at all. Then you would not know what the word "and" means, and you would not know that, if you have two sentences P and Q, then you can say or write "P and Q". As you learn English, at some point you would learn the meaning of the word "and" and then you would learn that when you have two sentences P and Q, then you can say or write "P and Q". (And I would even argue that this rule about that use of "and" is in fact what "and" means, but I will not pursue this now.) The point is: *there are* rules for using the word "and", and those rules have to be *learned*, and they only look obvious to you because you already learned them a long time ago and have grown accustomed to them.

What we are doing in Logic is **elucidating the laws of thought, making them explicit, bringing them to the surface, as it were**, so that we can, for example, pass them on from our minds to a computer: the computer does not "know" any of the things that you know, unless you tell the computer those things. And this applies even to the rules that you know so well that they are deeply embedded in your subconscious, so you take them for granted without even realizing that there is something to be known there.

Once you understand this, you will also see that **it is not an accident that modern Logic developed first, at the end of the 19th century and the beginning of the 20th century, and computers came into being soon afterwards.** □

20.5.2 Using a conjunction: another stupid but important rule

> **The rule for using a conjunction (Rule $\wedge_{use}$)**
>
> If P, Q are sentences, and you have proved $P \wedge Q$, then you are allowed to go to P, and you are also allowed to go to Q.

IMPORTANT REMARK. This looks like a very stupid rule. But you should reread the "Important Remark" on Page 359, where we talked about another "stupid rule", namely, Rule $\wedge_{prove}$. That remark also applies to Rule $\wedge_{use}$. □

20.6 Disjunctions ("$\vee$", i.e., "or")

The symbol

$$\bigvee$$

is the ***disjunction symbol***, and means "or".

So, for example,

- If P is the sentence

today is Friday

and Q is the sentence

today is Saturday

then "$P \vee Q$" stands for the sentence

today is Friday or today is Saturday.

- A sentence of the form $P \vee Q$ is a ***disjunction***.
- In a disjunction $P \vee Q$, the sentences P, Q are the ***disjuncts***.

20.6.1 Using a disjunction: the "proof by cases" rule

The rule for using a disjunction, that we are going to call "Rule $\vee_{use}$", as you may have guessed, is extremely important. It is also called the "proof by cases rule", and is one of the most widely used rules in theorem proving.

Before I state the rule, let us look at an example.

Example 48. Suppose you want to prove that

$$(\forall x \in \mathbb{R})(x \neq 0 \Longrightarrow x^2 > 0\,. \tag{20.27}$$

Then you could reason as follows. Since $x \neq 0$, there are two possibilities: $0 < x$ or $x < 0$. We consider each of these two possibilities separately.

First we assume that $\boxed{0 < x.}$

Then we use the fact that we can multiply both sides of an inequality by a positive number[7]. Since $0 < x$ (because we are assuming that $0 < x$), we can multiply both sides of "$0 < x$" by x, and get $x.0 < x.x$.

But $x \cdot 0 = 0$ by a theorem[8]

And $x \cdot x = x^2$. (This is because the definition of x^2 says that $x^2 = x \cdot x$.)

[7]This is one of the axioms of real number theory, that we will discuss later. Tha axiom says: $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})\Big((x < y \wedge 0 < z) \Longrightarrow xz < yz\Big)$.

[8]The theorem says that $(\forall x \in \mathbb{R})x.0 = 0$. This was proved earlier for integers, but the proof for real numbers is the same.

So $\boxed{0 < x^2}$.

Next we assume that $\boxed{x < 0.}$

Then we usethat axiom that says that we can add a real number to both sides of an inequality and the result is an inequality going in the same direction[9]. So we add $-x$ to both sides of "$x < 0$" and get $0 < -x$.

Then we use the axiom about multiplication of both sides of an inequality by a positive number. Since $-x$ is positive, because we have proved that it is (under the assumption that $x < 0$), we can multiply both sides of "$0 < -x$" by $-x$, and get $(-x).0 < (-x).(-x)$.

But $x \cdot 0 = 0$ by a theorem proved before.

And $(-x) \cdot (-x) = x \cdot x$.

So $0 < x \cdot x$.

And $x \cdot x = x^2$, by the definition of "square".

So $\boxed{0 < x^2}$ in this case as well.

So we have analyzed each of the two possibilities $0 < x$ and $x < 0$, and in each case we arrived a the same conclusion, namely, that $0 < x^2$.

Hence we have proved that $\boxed{\boxed{0 < x^2}}$.

What we have done in this example is this: we knew that a disjunction $A \vee B$ was true. (In our example, A was "$0 < x$" and B was "$x < 0$".) Then we proved that a ceartain conclusion C must hold if A is true, and also if B is true. (In our example, C was "$0 < x^2$".) Then we concluded that C must be true. And the reason is quite simple: one of A, B is true, and in either case C is true, so C is true.

This is exactly what the proof by cases rule says.

---

[9] Precisely, the axiom says: $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})\Big(x < y \Longrightarrow x + z < y + z\Big)$.

The rule for using a disjunction (Rule $\vee_{use}$, a.k.a. the proof by cases rule)

If P and Q are sentences, and you have proved $P \vee Q$ in a previous step, and then you prove another sentence R both assuming P and assuming Q, then you can go to R.

20.6.2 Proving a disjunction

The rule for proving a disjunction (Rule $\vee_{prove}$)

Suppose P and Q are sentences, and you want to prove $P \vee Q$. Here is what you can do. You look at the two possible cases, when P is true and when P is false. If P is true then of course $P \vee Q$ is true, so we are O.K. So all we have to do is look at the other case, when P is false, and prove that in that case Q is true.
So here is the rule:

I. If, assuming that P is false, you can prove Q, then you can go to $P \vee Q$.

II. If, assuming that Q is false, you can prove P, then you can go to $P \vee Q$.

20.7 Implications (“$\Longleftrightarrow$”, i.e., “if ... then”)

Implication: The symbol

$$\Longrightarrow$$

is the ***implication symbol***, and means “implies”.

A sentence “$P \Longrightarrow Q$” is read as

$$P \text{ implies } Q$$

or as

$$\text{If } P \text{ then } Q\,.$$

Then

- If P is the sentence

  Today is Friday

  and Q is the sentence

  Tomorrow is Saturday

  then "$P \Longrightarrow Q$" stands for the sentence

  If today is Friday then tomorrow is Saturday.

- A sentence of the form $P \Longrightarrow Q$ is an ***implication***, or a ***conditional sentence***.
- In a conditional sentence $P \Longrightarrow Q$, P is the ***premiss*** (or ***antecedent***), and Q is the ***conclusion*** (or ***consequent***.

20.7.1 The rule for using an implication

We now come to one of the most important rules in Logic: the rule for using an implication. For us, this rule will be called— guess what!—"Rule $\Longrightarrow_{use}$", but it also has a couple of much more impressive names: **Modus Ponens**, and **implication elimination**[10]

> **The rule for using an implication (Rule $\Longrightarrow_{use}$, a.k.a. *Modus Ponens*)**
>
> Suppose P, Q are sentences. Suppose you have the sentences $P \Longrightarrow Q$" and "P" in previous steps of your proof. Then you can go to Q.

Example 49. Suppose you know that "If you are a student then you are entitled to a discount" and you also know that you are a student. Then you can conclude that you are entitled to a discount.

[10] "Modus Ponens" is an abbreviation of "modus ponendo ponens", which is Latin for "the way that affirms by affirming".

20.7.2 The "for all...implies" combination

One of the most important and widely used combinations of moves in proofs is what we may call ***the "for all...implies" combination***.

It works like this:

- First, you bring into your proof a statement S of the form "for every x of some kind, if something happens then something else happens". That is, $(\forall x)(A(x) \Longrightarrow B(x))$, or
$$(\forall x \in S)(A(x) \Longrightarrow B(x))\,. \tag{20.28}$$
- Then, you bring into your proof an object a for which you know that this object satisfies Property A, that is, you know that
$$A(a)\,. \tag{20.29}$$
- Then you derive the conclusion that $B(a)$ is true, in two steps:

  Step 1: Use the specialization rule to go from (20.28) to
$$A(a) \Longrightarrow B(a)\,. \tag{20.30}$$
  Step 2: Use Modus Ponens to go from (20.30) and (20.29) to
$$B(a)\,. \tag{20.31}$$

This combination is used all the time in proofs. The reason is that many theorems in Mathematics are of the form: "whenever something is true of an object, then something else is also true of that object", that is
$$(\forall x)(A(x) \Longrightarrow B(x))\,. \tag{20.32}$$
And what you often do in proofs is take one of those theorems and apply it to a particular situation. And this is exactly what the "for all...implies" combination does.

Here are some examples:

1. Take the statement that "Every positive real number has a real square root", which translates into
$$(\forall x \in \mathbb{R})(x > 0 \Longrightarrow (\exists y \in \mathbb{R})y^2 = x)\,.$$
   This is exactly of the form (20.32), with "$x > 0$" in the role of $A(x)$, and "$(\exists y \in \mathbb{R})y^2 = x$" in the role of $B(x)$.

Then you can prove that 2 has a square root, by applying the "for all ... implies" combination, with $a = 2$, and getting "$(\exists y \in \mathbb{R})y^2 = 2$".

2. Suppose you know that "If x is a positive real number then $x + \frac{1}{x} \geq 2$", that is, in formal language,

$$(\forall x \in \mathbb{R})(x > 0 \Longrightarrow x + \frac{1}{x} \geq 2)\,.$$

(We will prove this later.) Suppose you have a real number a, and have proved that a is positive (that is, $a > 0$). Then you can draw the conclusion that $a + \frac{1}{a} \geq 2$ by using the "for all...implies" combination, as follows:

1. $(\forall x \in \mathbb{R})(x > 0 \Longrightarrow x + \frac{1}{x} \geq 2)$ [Fact proven before]
2. $a > 0$.[Known]
3. $a > 0 \Longrightarrow a + \frac{1}{a} \geq 2$.[Rule $\forall_{use}$, from Step 1]
4. $a + \frac{1}{a} \geq 2$.[Rule $\Longrightarrow_{use}$, from Steps 2,3]

20.7.3 Proving an implication

The rule for proving an implication (Rule $\Longrightarrow_{prove}$)

Suppose P, Q are sentences. Suppose you start a proof with "Assume P", and you prove Q. Then you can go to $P \Longrightarrow Q$.

Example 50. Say you are a Martian who just landed on Earth, you know nothing about the days of the week, and you want to prove that to your own satisfaction that "If today is Friday then tomorrow is Saturday". To apply Rule $\Longrightarrow_{prove}$, you would begin by "assuming that today is Friday." This means that you would imagine that today is Friday, and see what would happen in that case. For example, you could go to a public library and look at lots of newspapers published on a Friday, and you would see that every time such a paper talks about the following day it says something like "tomorrow is Saturday." Then you would be reasonably confident that the sentence "If today is Friday then tomorrow is Saturday" is true. And it would not matter whether today is Friday or not. $\square$

20.7.4 The connectives "$\wedge$" and "$\Longrightarrow$" are very different

Students sometimes think that "If P then Q" is basically the same as "P and Q", or "P then Q". But this is very wrong and it important that you should understand the difference between "P and Q" and "If P then Q".

Take, for example, the sentences

Today is Friday and tomorrow is June 12.

and

If today is Friday then tomorrow is June 12.

Using "P" to represent the sentence "Today is Friday" and "Q" to represent the sentence "Tomorrow is June 2", the first sentence is $P \wedge Q$, and the second one is $P \Longrightarrow Q$.

What conditions have to be satisfied for $P \wedge Q$ to be true? What conditions have to be satisfied for $P \Longrightarrow Q$ to be true?

The sentence $P \wedge Q$ is true if both P and Q are true. In our example, the only way the sentence "Today is Friday and tomorrow is June 12" can be true is if today is Friday and tomorrow is June 12, So ***the sentence "Today is Friday and tomorrow is June 12" is true if today is Friday June 11, and in no other case***.

On the other hand, ***The sentence $P \Longrightarrow Q$ when Q is true, and also when P is false. And if neither one of these conditions hold (that is, if Q is false and P is true) then $P \Longrightarrow Q$ is false***. So, in our example, the only possible situation when "If today is Friday then tomorrow is June 12" would be false is if today is Friday but tomorrow is not June 12. So ***he sentence "If today is Friday then tomorrow is June 12" is true if today is not Friday, is also true if tomorrow is June 12, and is false if today is Friday but tomorrow is not June 12***.

We can summarize these observations by means of the following "truth tables" for the connectives "$\wedge$" and "$\Longrightarrow$":

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \Longrightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

The first table gives you the truth value[11] of $P \wedge Q$ in terms of the truth values of P and Q, and the second table gives you the truth value of $P \Longrightarrow Q$ in terms of the truth values of P and Q.

Notice that ***what makes the truth tables for "wedge" and "$\Longrightarrow$" is the last two lines***. In particuler:

$$\boxed{\boxed{\boxed{\begin{array}{l} P \Longrightarrow Q \textbf{\textit{ is always true when }} Q \textbf{\textit{ is true,}} \\ \textbf{\textit{no matter whether }} P \textbf{\textit{ is true or false}}. \end{array}}}}$$

and

$$\boxed{\boxed{\boxed{\begin{array}{l} P \Longrightarrow Q \textbf{\textit{ is always true when }} P \textbf{\textit{ is false,}} \\ \textbf{\textit{no matter whether }} Q \textbf{\textit{ is true or false}}. \end{array}}}}$$

So for example, the following sentences are true:

- If the Earth is a planet then 3 is a prime number.
- If the Earth is a comet then 3 is a prime number.
- If the Earth is a comet then 6 is a prime number.

The first one and the second one are true because the conclusion (that is, "3 is a prime number) is true. . (It does not matter, for the second sentence, that the premiss—"the Earth is a comet"— is false.)

And the second one and third one are true because the premiss ("the Earth in a comet" is false. (It does not matter whether for the second sentence, that the conclusion—"6 is a prime number"— is false.)

[11]Every sentence, when used correctly, has a ***truth value***: the truth value is T is the sentence is true, and F is the sentence is false. For example: the truth value of "$3 > 5$" is F, the truth value of "$3 < 5$" is T. How about the truth value of "$x < 5$". If you tell me that $x < 5$ without having told me who x is, then I do not knwo the truth value of "$x < 5$". But this would be an incorrect us of "$x < 5$". If you were writing a proof, then you could never have "$x < 5$" as one of the steps, unless you have told the reader before, in some previous step, who x is, and once you have done that, the truth value of "$x < 5$" would be known. For example, if you said in a previous step "Let $x = \frac{1+\sqrt{5}}{2}$", then I would know that "$x < 5$" is true. (Proof: $\sqrt{5} < 5$. So $1 + \sqrt{5} < 6$. So $\frac{1+\sqrt{5}}{2} < 3$. Hence $\frac{1+\sqrt{5}}{2} < 5$. So $x < 5$.)

On the other hand, the sentence "If the Earth is a planet then 6 is a prime number" is false, because the premiss ("The Earth is a planet") is true, but the conclusion ("6 is a prime number") is false.

20.7.5 Isn't the truth table for $\Longrightarrow$ counterintuitive?

Students often ask questions about the implication connective $\Longrightarrow Q$ and in partuclar about the truth table for the implication.

One often raise question is "how can '$P \Longrightarrow Q$' be true if P and Q have nothing to do with each other?".

For example, we said that the sentence "If the Earth is a planet then 3 is a prime number" is true, but what does the fact that the Earth is a planet have to do with 3 being a prime number? That sounds like a good question, but let us think about it. I suggest that you do do this:

Think of "$P \Longrightarrow Q$" as saying "it does not happen that P is true without Q also being true".

In other words: what "$P \Longrightarrow Q$" does is exclude the possibility that you might ever run into a "bad situation", menaing, "a situation where P is true but Q is not". And this is the only possibilty excluded the implication. So, in particular,

- if P is false then you will not be in a bad situation, so "$P \Longrightarrow Q$" is true.
- if Q is true then you will not be in a bad situation, so "$P \Longrightarrow Q$" is true.

Once you understand this, you will see that it does not matter very much whether P and Q have something to do with each other. Maybe $P$and Q are totally unrelated, but if, for example, they both happen to be true then "$P \Longrightarrow Q$" is true. And also, "$P \Longrightarrow Q$" will be true if both P and Q are false, or if P is false and Q is true.

Example 51. Suppose a street sign says:

IF YOU ARE DRIVING AT MORE THAN 25MPH YOU WILL GET A FINE.

Supoose you want to prove to a friend of yours that the municipal government that put up the sign isn't really enforcing its own rule. What do you have to do to prove this?

Let "P" represent the premiss, i.e., "you are driving at more than 25mph", and let "Q" represent the conclusion, that is, "you will get a fine". Then the street sign asserts the implication "$P \Longrightarrow Q$".

Certainly,

- If you find someone driving at 20mph, that will do nothing to prove your case. ***That's because in that case the implication "$P \Longrightarrow Q$" is true, according to the truth table for the implication.*** It does not matter whether that driver got a fine or not[12].

- If you find someone who got a fine, that will do nothing to prove your case. ***That's because in that case the implication "$P \Longrightarrow Q$" is true, according to the truth table for the implication.*** It does not matter whether that driver was driving at more than 25mph or not.[13].

- The only way to prove that the injunction in the street sign is not being enforced is to find cases of drivers that were driving at more than 25mph but did not get a fine. ***That's because the onlt case when the implication "$P \Longrightarrow Q$" is false, according to the truth table for the implication, is when the premiss is true but the conclusion is false.***

Example 52. Alice is a cashier at a department store, and she has to follow the rule that

[12]The driver may have been given a fine for some other reason, e.g., using a cell phone while driving.

[13]The driver may have been driving at 20mph but may have been given a fine for some other reason, e.g., using a cell phone while driving.

IF A CUSTOMER PAYS CASH FOR A PURCHASE THEN ALICE HAS TO PUT THE MONEY SHE COLLECTED IN A DRAWER.

Suppose you are a detective and you want to prove that Alice is not obeying the rule. What do you have to do?

- If you find a situation when there was not customer at all, or there was customer that did not pay cash, then that will do nothing prove your case. ***That's because in that case the implication "$P \Longrightarrow Q$" is true, according to the truth table for the implication.*** It does not matter whether Alice put money is the drawer or not[14].

- If you find a situation where Alice put cash in the drawer even though she did not collect any money from a customer, then that will do nothing to prove your case. ***That's because in that case the implication "$P \Longrightarrow Q$" is true, according to the truth table for the implication.*** It does not matter that there was no customer poaying cash[15].

- The only way you can prove that Alice is violating the rules is by showing that a customer paid cash but Alice did notput themoney in the drawer. ***That's because the only case when the implication "$P \Longrightarrow Q$" is false, according to the truth table for the implication, is when the premiss is true but the conclusion is false.***

Example 53. Suppose you have a natural number n, but you do not know which number it is. (For example, maybe someone gave you a sealed envelope containing a card where the number is written. So the number is there, it's a fixed number, but you just do not knwo which specific number it is.)

Suppose you are asked to prove that

[14]Why would Alice have put money in the drawer if she did not collect any cash from the customer? Who knows?

[15]Again, why would Alice put money in the drawer even if she did not collect the money from a customer? Who knows? And who cares? The point is: ***even if she put money in the drawer when there had been no customer that paid her the money, so P was false but Q was true, she did not violate the rules.***

(*) If n is even then n^2 is divisible by 4.

Then you could ask: could (*) possibly be false? Could there be a possible value of n for which (*) is false. (Remember that you do not know who n is. So if you want be able to assert for sure that (*) is true you have to consider all possible values of n. If you find one value of n for which (*) is not true, then you cannot be sure that n is true, because the number that you have in the envelope could be the one you have found, the one for which (*) is false. But if you can make sure that no such number exists, then you can be sure that (*) is true, even though you do not know who n is.)

What would have to happen for (*) to be false? Well, according to our truth table, the only case when the implication (*) is false is when the premiss is true but the conclusion is not. So to make sure that (*) is true, you have to consider numbers n that are even, because if n is not even then (*) is true. You indicate that you are going to do that by writing:

Assume that n is even.

(In other words: ***you are allowed to assume that*** n ***is even because if*** n ***is not even then (*) is automatically true thanks to the truth table for the implication.***)

And then you move on to prove that n^2 is divisible by 4. (Since n is even, we can pick a natural number k such that $n = 2k$. Then $b^2 = 4k^2$, so n^2 is divisible by 4.)

And now you can be sure that (*) is true. The number n is even or odd, but in either case (*) is true, even though in each case it's true for a different reason: if n is not even, then (*) is true because of the truth table for the implication, and if n is even then (*) ia true because in that case we have proved that the conclusion (that is, "n^2 is divisble by 4") must be true.

Finally, we have prove that (*) must be true for any natural number, because we have proved for n, but n could be any number. So we can conclude that

$$(\forall n \in \mathbb{N})\Big(n \text{ is even} \Longrightarrow n^2 \text{ is divisible by } 4\Big)\,,$$

or, if you prefer,

$$(\forall n \in \mathbb{N})\Big(2|n \Longrightarrow 4|n^2\Big)\,.$$

So we can structure our proof as follows:

THEOREM. $(\forall n \in \mathbb{N})\Big(2|n \Longrightarrow 4|n^2\Big)\,.$

PROOF We want to prove that $(\forall n \in \mathbb{N})\Big(2|n \Longrightarrow 4|n^2\Big)$.

Let $n \in \mathbb{N}$ be arbitrary.

We want to prove that $2|n \Longrightarrow 4|n^2$.

Assume that $2|n$.
Then $(\exists k \in \mathbb{N})n = 2k$.
Pick one such k and call it k_*.
Then $k_* \in \mathbb{N}$ and $n = 2k_*$.
Then $n^2 = (2k_*) \cdot (2k_*) = 4k_*^2$.
Let $q = k_*^2$.
Then $n^2 = 4q$.
So $(\exists k)n^2 = 4k$.
Hence $4|n^2$.

We have proved that $4|n^2$ assuming that $2|n$. Hence

$$2|n \Longrightarrow 4|n^2\,.$$

We have proved that $2|n \Longrightarrow 4|n^2$ for an arbitrary n. Therefore

$$(\forall n \in \mathbb{N})\Big(2|n \Longrightarrow 4|n^2\Big)\,.$$

Q.E.D.

I hope that these remarks will suffice to clarify they way implication works. Implication will be discussed in great detail later.

20.8 Biconditionals ("$\Longleftrightarrow$", i.e., "if and only if")

The ***biconditional*** is the symbol

$$\Longleftrightarrow .$$

It is a *binary connective*, like $\wedge$, $\vee$, and $\Longrightarrow$. That means that $\Longleftrightarrow$ ***can be used to connect two sentences***.

If P and Q are sentences, the sentence "$P \Longleftrightarrow Q$" is read as

$$\boxed{P \text{ if and only if } Q}$$

or

$$\boxed{P \text{ is equivalent to } Q}\,.$$

And mathematicians often use "iff" as shorthand for "if and only if", so they write "P iff Q."

$$\boxed{P \text{ iff } Q}\,.$$

The precise meaning of "equivalence" will be explained later. But, if you want to know right away what it means, it's very simple:

> *When you know that P is equivalent to Q then you can pass freely from P to Q. That is, if you know that P is true then you can write Q, and if you know that Q is true then you can write P.*
> *So for all practical purposes if "$P \iff Q$" is true then P and Q are interchangeable.*

20.8.1 The meaning of "if and only if"

> *You should think of "P iff Q" as meaning*
>
> $$(P \iff Q) \wedge (Q \iff P)\,.$$

That is, "$P \iff Q$" means[16]

[16] *This note is only for philosophically minded nitpickers.* What does "means" mean? The point of view adopted here is that *the meaning of a word, phrase or symbol consists of the rules for the use of that word, phrase or symbol.* For example, the meaning of "and" is the specification that if P, Q are two sentences, then (i) if you have "P and Q" you can go to P and you can go to Q, and (ii) if you have P and you have Q then you can go to "P and Q." That is, *the meaning of "and" is captured by Rules $\wedge_{use}$ and $\wedge_{prove}$.* Naturally, this does not cover all the uses of "and" in our culture, such as, for example, to indicate a progression (as in "this is getting better and better"), or to indicate a causal relation, (as in "do that and I'll hit you"), or the literary use full of nuances (as 'in 'tomorrow and tomorrow and tomorrow"). And, most importantly for us, it does not cover the use of "and" to connect *nouns*, as in "slings and arrows". But it's what "and" means in logic and mathematics. If you want to program a computer so that it will know what "and" means,

$$\text{If } P \text{ then } Q \text{ and if } Q \text{ then } P\,,$$

or

$$P \text{ implies } Q \text{ and } Q \text{ implies } P\,.$$

In order to make this true, we will choose the rules for proving and using biconditional sentences as follows:

- *To prove "$P \iff Q$" you do exactly the same thing that you would do to prove $(P \iff Q) \wedge (Q \iff P)$.*
- *To use "$P \iff Q$" you do exactly the same thing that you would do to use $(P \iff Q) \wedge (Q \iff P)$.*

So, for example, suppose you want to prove that

$$(\forall x \in \mathbb{R})\Big(x^2 = 4 \iff (x = 2 \vee x = -2)\Big)\,. \tag{20.33}$$

Then you would start by introducing into your proof an arbitrary real number called x, and then you would prove that

$$(x^2 = 4 \iff (x = 2 \vee x = -2)\,. \tag{20.34}$$

And to prove (20.34), which is an "iff" sentence, you would prove both implications $x^2 = 4 \Longrightarrow (x = 2 \vee x = -2)$ and $(x = 2 \vee x = -2) \Longrightarrow x^2 = 4$. (The proof of these two sentences is very simple: to prove that $x^2 = 4 \Longrightarrow (x = 2 \vee x = -2)$, you use the fact that a positive real number r cannot have more than two square roots[17]. Since 2 and -2 are two distinct square roots of 4, there cannot be a third square root. So, if $x^2 = 4$, so x is a square root of 4, it follows that x must be 2 or -2. So $x^2 = 4 \Longrightarrow (x = 2 \vee x = -2)$. To prove the other implication, i.e., that $(x = 2 \vee x = -2) \Longrightarrow x^2 = 4$, just observe that if $x = 2$ then $x^2 = 4$, and if $x = -2$ then $x^2 = 4$ as well,)

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you have to tell the computer how to use "and". And this amounts to programming the computer to use rules $\wedge_{use}$ and $\wedge_{prove}$. And you don't need to tell the computer anything else. A similar situation arises with the biconditional. A computer that "knows" the rules $\iff_{use}$ and $\iff_{prove}$ "knows" all it needs to know to work with the biconditional, and for that reason I believe that knowing the meaning of "$\iff$" amounts to knowing the two rules for working with it.

[17]This was proved in the notes for Lectures 2,3,4 but, just in case, here is a quick proof: suppose r has three distinct square roots a, b, c. Then $a^2 = r$, $b^2 = r$ and $c^2 = r$. Hence $a^2 - b^2 = 0$. So $(a - b)(a + b) = 0$. Therefore $a - b = 0$ or $a + b = 0$. Since a and b are different, it cannot be the case that $a - b = 0$, so $a + b$ must be zero, and then $b = -a$. Now we can use exactly the same argument with c instead of b, and conclude that $c = -a$. But then $c = b$, contradicting the fact that $b \neq c$.

20.8.2 The rules for proving and using biconditionals

Now let us state explicitly the rules for proving and using biconditional sentences.

As I explained in the previous subsection, *these rules are designed so as to make "$P \iff Q$" mean precisely what we want it to mean, that is "$(P \Longrightarrow Q) \wedge (Q \Longrightarrow P)$".*

The rules are as follows.

Rule $\iff_{prove}$

If P, Q are sentences, and you have proved the sentences

$$P \Longrightarrow Q$$

and

$$Q \Longrightarrow P\,,$$

then you can go to

$$P \iff Q\,.$$

Rule $\iff_{use}$

If P, Q are sentences, and you have proved the sentence

$$P \iff Q\,,$$

then you can go to

$$P \Longrightarrow Q$$

and you can also go to

$$Q \Longrightarrow P\,.$$

20.9 The other six rules

So far I have given you eight rules, two for each of the connectives $\wedge$, $\vee$, $\Longrightarrow$, and $\iff$.

In addition, there are six more rules that we have already discussed:

1. Rule $\forall_{prove}$, the rule for proving a universal sentence. (This rule is sometimes called "universal generalization".)

2. Rule $\forall_{use}$, the rule for using a universal sentence. (This rule is sometimes called the "specialization rule".)

3. Rule $\exists_{prove}$, the rule for proving an existential sentence.. (This rule is sometimes called the "witness rule".)

4. Rule $\exists_{use}$, the rule for using a universal sentence. (This rule is sometimes called the "existential specialization rule".)

5. The proof by contradcition rule.

6. Rule SEE substitution of equals for equals.

So now we have all fourteen rules.

20.10 Are the logical rules hard to understand and to learn and remember ?

Most of the logical rules are very simple and easy to remember. For example,

- The rules for using and proving $\wedge$ sentences are so stupid that you might object to having them because they are so obvious, but you certainly cannot find it hard to understand them.
- The rules for using and proving universal sentences are also natural:
  - if you know that all the items in this store cost 1 dollar, and you pick an item in this store, you can be sure that it costs 1 dollar. That's all that Rule $\forall_{use}$ says.
  - if you prove that a schmoo must be green, without using any information about that schmoo other than the fact that it is a schmoo, then you can conlude that all schmoos are green. And that's= all that Rule $\forall_{prove}$ says.
- And the rules for using and proving existential sentences are natural as well:
  - if you know that somewhere in this store there is a schmoo, then you can go and get a schmoo and call it any way you want, for example "my woderful schmoo". That's all that Rule $\exists_{use}$ says.
  - if you find a schmoo, then you can conclude that schmoos exist. And that's all that Rule $\exists_{prove}$ sats.

20.10.1 Proofwriting and rules for proofs

Writing proofs is like playing chess, checkers, or some other board game.

- There are rules that tell you which moves are allowed. (Notice that the rules for proofs never say "you ***have*** to do this". They say "you ***are allowed*** to do this". It's exactly like the moves you are allowed to make in a board game.)
- You have to obey the rules all the time.
- If you cheat, by violating the rules once, then you are out of the game.
- If you know how to play, you will never make a move that violates the rules.
- Once you know the moves, then the hard part begins: you have to figure out how to choose which moves to make in order to win. And that is where proofwriting becomes difficult and challenging: some people are better than others at figuring out how to win.
- From 1637 until 1995, many mathematicians tried very hard to prove Fermat's last theorem. Finally, Andrew Wiles suceeded in doing it in 1995.
- But the proofs we do in this course are not that hard.

20.11 The logical rules, besides being easy to understand and learn, will help you figure out how to prove what you want

Here is an example:

Suppose you want to prove by induction that every tournament has a top player. Then:

- In order to do induction , you need to express what you want to prove as "$(\forall n \in \mathbb{N})P(n)$". because induction is a method for proving sentences of the form "$(\forall n \in \mathbb{N})P(n)$".
- The most obvious way to do this is to let n be the number of players so that the statement "every tournament has a top player" is rewritten as "for eevry $n \in \mathbb{N}$, every tournament with n players has a top player"
- So you let $P(n)$ be the statement "every tournament with n players has a top player".
- And now you can set out to prove $(\forall n)P(n)$ by induction.
- The base step is easy.
- For the inductive step, you know that you have to prove that

$$(\forall n \in \mathbb{N})(P(n) \Longrightarrow P(n+1)) .$$

- Then Rule $\forall_{prove}$ tells you that it would be a good idea to start with "let n be an arbitrary natural numebr", and try to prove that $P(n) \Longrightarrow P(n+1)$.
- Now that you are trying to prove that $P(n) \Longrightarrow P(n+1)$, Rule $\Longrightarrow_{prove}$ ttell you that it would be a good idea for you to assume $P(n)$ and try to prove $P(n+1)$.
- Now that you are trying to prove $P(n+1)$, which is a universal sentence ("every tournament with $n+1$ players has a top player"), Rule $\forall_{prove}$ tells you that it would be a good idea start with "let T be an arbitrary tournament with $n+1$ players" and try to prove that T has a top player.

- In order to prove that T has a top player, you observe that "T has a top player" is an existential statement ("there exists a player of T that is a top player"). So Rule $\exists_{prove}$ tells you that it would be a good idea to find a top player of T.

- Since the only thing we know that enables us to find top players is $P(n)$, we should use $P(n)$.

- Since $P(n)$ says "every tournament with n players has a top player", the only way we can use this is to apply Rule $\forall_{use}$ to some tournament S with n players and get a top player of that tournament.

- So we need to produce somehow a tournament S with n players, so that we can invoke $P(n)$ and use Rule $\forall_{use}$ to get a topl player q of S.

- Since we have oour tournament T with $n+1$ players, and we need to get a tournament S with n players, the most natural thing to do is to start with T and remove one player.

- If we call this player p, then we should let S be the tournament obtained from T by removing this player p.

- Then $P(n)$ will enable us to get a top player q of S.

- And, finally, we find ourselves n the following situation:

    a. We have this tournament T with $n+1$ players and we are trying to prove that it has a top player.
    b. We also have the tournament S with n players that we have constructed from T by removing a player called p.
    c. We have a top player of S, called q.
    d. We are trying to prove that T has a top player.

- Then a natural candidate for the role of top player of T is q.

- So we should try to prove that q is a top player of T.

- And, up to this point, we haven't really done any hard thinking. The rules of logic guided us and told us what to do. We closed our eyes and followed them.

- And now, and only now, the hard thinking starts.
- We have to figure out how to prove that q is a top player of T.
- And we haven't yet said how to choose p.
- It may happen that we can prove that q is a top player of T without using any information at all about p. (That is, p could be an arbitrary player of T.)
- Or it may happen that we can prove that q is a top player of T only if p has some special features.
- And in that case we need to prove that a player p that has those speical features exists. (And if such a player exists, then we will be able to pick one using Rule ($\exists_{use}$.)

This is as far as pure logic has led us. And is it quite far. Now you have to do some hard thing and figur eout how to pick p nd how to prove that this choice of p works. .

Problem 44. ***Do (or redo, if you have already done it)*** *the "top player" problem. (That is, problem 11 on page 126 of the book.)*

First, you should read carefully the following things:

1. *The long hint that was given as part of the list of problems for homework assignment No. 7.*
2. *Sections 20.2.7 and 20.11 of these notes.*

And then ***you should write your proof following step by step the guidelines explained in Section 20.11.***

If you have already submitted this problem, then please do it again, making a special effort to write correctly, with precision and clarity. In particular, your proof should explain clearly how the player p is chosen, and should prove that, with that choice of p, player q is a top player.

This problem will be graded taking seriously into account the clarity, precision and correctness of the writing.

If you cannot figure out how to choose player p, send me e-mail and I will tell you[18] how to do it. □

[18] If all goes well, I will be reading and answering e-mail as of Sunday, November 12. Unfortunately, I will not able to read or answer e-mail on Saturday.