# MATHEMATICS 361 — FALL 2019
# SET THEORY

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## HOMEWORK ASSIGNMENT NO. 3, DUE ON THURSDAY, SEPTEMBER 26

*These problems are about "pure set theory"; that is, we assume there are no atoms, so everything is a set. Hence "$\forall x$" means "for every set $x$", and "$\exists x$" means "there exists a set $x$ such that".*

1. Book, Pages 38, 39, Exercises 1, 2, 3, 4, 5.

2. Consider the following theorem and its proof:

   **Theorem.** *If $A, B$ are sets and $A \times B = B \times A$ then $A = B$.*

   *Proof.* If $x \in A$ and $y \in B$ then $< x, y > \in A \times B$; since $A \times B = B \times A$, it follows that $< x, y > \in B \times A$; so $x \in B$ and $y \in A$. This shows that if $x \in A$ then $x \in B$, so $A \subseteq B$, and that if $y \in B$ then $y \in A$, so $B \subseteq A$. Hence $A = B$. **Q.E.D.**

   Actually, ***the theorem, as stated, is false, so the proof must be wrong***.

   i. ***Prove*** that the theorem is false, by giving a counterexample.

   ii. ***Explain*** why the proof is wrong, that is, find the step or steps that are invalid.

   iii. ***Fix*** the theorem, by adding an extra condition to the hypotheses of the theorem that makes it true.

   iv. ***Give a correct proof*** of the true theorem.

NOTE: In case you don't understand what I mean by "fixing" a theorem statement, here is an example, that should illustrate what I mean.

Suppose I give you the theorem that *if $a, b, c$ are real numbers and $ab = ac$ then $b = c$.* This theorem is also false. Suppose I ask you the same questions that I asked here.

To prove that the theorem is false, you must give a counterexample. Here is a counterexample: take $a = 0$, $b = 1$, $c = 2$. Then $ab = ac$, but $b$ is not equal to $c$.

To find the mistake in the proof, you have to look at the proof. If you do that, I am sure you will find a step that involves dividing both sides by $a$, and this step is invalid because we don't know that $a \neq 0$.

To fix the theorem, add the condiition that $a \neq 0$. Then you get a true theorem: *if* $a, b, c$ *are real numbers such that* $a \neq 0$*, and* $ab = ac$*, then* $b = c$*.*

Finally, the correct proof of the true theorem is easy: since $a \neq 0$, $a^{-1}$ exists. Multiply both sides of "$ab = ac$" by $a^{-1}$, and you get $b = c$. **Q.E.D.**

In the set theory problem here, I am asking you to do something similar.