MATHEMATICS 361 — FALL 2019
SET THEORY
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HOMEWORK ASSIGNMENT NO. 6, DUE ON TUESDAY, NOVEMBER 12

These problems are about "pure set theory"; that is, we assume there are no atoms, so everything is a set. Hence "$\forall x$" means "for every set x", and "$\exists x$" means "there exists a set x such that".

PRACTICE PROBLEMS FOR YOU TO DO:

- Book, pages 61-62, exercises 32 to 42.
- Book, page 101, exercises 1 to 9.
- Book, page 111, exercises 10 to 14.
- Book, page 120-121, exercises 15 to 22.
- **Problem I**: Assume that $f : A \mapsto B$. Define $\sim_f$ to be the set

$$\left\{ \langle x, y\rangle \middle| f(x) = f(y) \right\}.$$

  1. ***Prove*** that $\sim_f$ is an equivalence relation on A. (HINT: This is a trivial special case of Exercise 36.)
  2. ***Prove*** that if $C \neq \emptyset$ and $g : A \mapsto C$ is a function, then there exists a function $h : B \mapsto C$ such that $h \circ f = g$ if and only if

$$\forall x\, \forall y \Big(x \sim_f y \Longrightarrow g(x) = g(y) \Big) \qquad (1)$$

- **Problem II**: ***Prove*** that if R is an equivalence relation on a set A then there exist a set B and a function $f : A \mapsto B$ such that $R = \sim_f$.
- **Problem III**: *This problem is about the "analogous results" mentioned in the statement of Theorem 3Q.* If R is an equivalence relation on a set A, and $f : A \times A \mapsto A$, then we say that <u>f is compatible with R</u> if

$$\forall x\, \forall x'\, \forall y\, \forall y' \Big((x \in A \,\&\, x' \in A \,\&\, y \in A \,\&\, y' \in A \,\&\, xRx' \,\&\, yRy') \Longrightarrow f(x, y) R f(x', y') \Big). \qquad (2)$$

Prove that if R is an equivalence relation on A and $f : A \times A \mapsto A$, then

(i) there exists a function $\hat{f} : A/R \times A/R \mapsto A/R$ such that

$$\hat{f}([x]_R.[y]_R) = [f(x,y)]_R \qquad \text{for all } x, y \in A \tag{3}$$

if and only if f is compatible with R.

(ii) if a function $\hat{f}$ such that (3) holds, then $\hat{f}$ is unique.

PROBLEMS FOR YOU TO HAND IN:

1. Problems I, II, and III.
2. Book, pages 61-62, Problems 36, 37, 38, 39, 40.
3. Book, page 101, Exercises 1, 3, 4.
4. Book, page 111, Exercise 14.
5. Book, page 120, Exercises 15, 16, and 22. (For exercise 22, make sure that you use the definition of "$-x$", for a real number x, given on page 117 —that is, $-x = \{r \in \mathbb{Q} \big| \exists s(s \in \mathbb{Q} \,\&\, s > r \,\&\, -s \notin x \big\})$— and the definition of "$|x|$" given on page 118 —that is, $|x| = x \cup -x$ — .)