MATHEMATICS 361 — FALL 2019

SET THEORY

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HOMEWORK ASSIGNMENT NO. 7, DUE ON THURSDAY, NOVEMBER 21

This homework assignment consists of four problems.

These problems are about "pure set theory"; that is, we assume there are no atoms, so everything is a set. Hence "$\forall x$" means "for every set x", and "$\exists x$" means "there exists a set x such that".

In the first two problems I present an alternative construction of the real numbers, as equivalence classes of Cauchy sequences of rationals, rather than as Dedekind cuts. Since several students are probably not familiar with Cauchy sequences, I am using instead a special class of Cauchy sequences, that I call "superCauchy sequences", because these are sufficient for our purposes, and are much easier to work with.

Recall that a sequence is a function whose domain is ω. If s is a sequence, then instead of writing $s(n)$ for the unique x such that $\langle n, x\rangle \in s$, we write s_n. (Then $s = \left\{ \langle n, s_n\rangle \,\middle|\, n \in \omega \right\}$.)

A sequence of rational numbers is a sequence s such that $s_n \in \mathbb{Q}$ for every $n \in \omega$.

If s is a sequence of rational numbers, we say that s is superCauchy if there exists a C such that

$$C \in \mathbb{Q} \,\&\, C > 0 \,\&\, \forall n \big(n \in \omega \Longrightarrow |s_{n+1} - s_n| \leq \frac{C}{2^n}\big)\,.$$

We let $SC(\mathbb{Q})$ be the set of all superCauchy sequences of rational numbers.

The sum $s+t$, the difference $s-t$, and the product $s \cdot t$ of two sequences s, t of rational numbers are defined as follows:

$$\begin{aligned} s+t &= \left\{ \langle n, s_n + t_n\rangle \,\middle|\, n \in \omega \right\}, \\ s-t &= \left\{ \langle n, s_n - t_n\rangle \,\middle|\, n \in \omega \right\}, \\ s \cdot t &= \left\{ \langle n, s_n \cdot t_n\rangle \,\middle|\, n \in \omega \right\}. \end{aligned}$$

A sequence $s \in SC(\mathbb{Q})$ is <u>null</u> if there exists a C such that

$$C \in \mathbb{Q} \,\&\, C > 0 \,\&\, \forall n \Big(n \in \omega \Longrightarrow |s_n| \leq \frac{C}{2^n} \Big) .$$

We let $SC(\mathbb{Q})_0$ be the set of all null superCauchy sequences of rational numbers.

We declare two superCauchy sequences s_1, s_2 to be <u>equivalent</u>, and write "$s_1 \sim s_2$", if $s_1 - s_2 \in SC(\mathbb{Q})_0$. In other words, $\sim$ is the relation

$$\Big\{ \langle s_1, s_2 \rangle \,\Big|\, \langle s_1, s_2 \rangle \in SC(\mathbb{Q}) \times SC(\mathbb{Q}) \,\&\, s_1 - s_2 \in SC(\mathbb{Q})_0 \Big\} .$$

Problem 1.

1. ***Prove*** that the sum and the product of two superCauchy sequences of rational numbers is a superCauchy sequence of rational numbers.

2. ***Prove*** that if s, t are null superCauchy sequences of rational numbers then $s+t$ and $s-t$ are null superCauchy sequences of rational numbers.

3. ***Prove*** that if s is a null superCauchy sequence of rational numbers and t is a superCauchy sequence of rational numbers then $s \cdot t$ is a null superCauchy sequence of rational numbers[1].

4. ***Prove*** that $\sim$ is an equivalence relation on $SC(\mathbb{Q})$.

5. ***Prove*** that the addition function $+ : SC(\mathbb{Q}) \times SC(\mathbb{Q}) \mapsto SC(\mathbb{Q})$, the subtraction function $- : SC(\mathbb{Q}) \times SC(\mathbb{Q}) \mapsto SC(\mathbb{Q})$, and the multiplication function $\cdot : SC(\mathbb{Q}) \times SC(\mathbb{Q}) \mapsto SC(\mathbb{Q})$ are compatible[2] with the equivalence relation $\sim$, in the sense of Problem III of Homework 6.

6. ***Conclude*** from the previous result that the sum, difference, and product of two equivalence classes $a, b \in SC(\mathbb{Q})/\sim$ are well defined. (That is, we can define $a+b$, $a-b$, and $a \cdot b$ by letting $a+b = [s+t]_\sim$, $a-b = [s-t]_\sim$, and $a \cdot b = [s \cdot t]_\sim$, where s is any member of a and t is any member of b.) $\square$

In view of the results of Problem 1, we can formulate the following definition.

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[1]For those who are familiar with rings and ideals, I am asking you to prove that $SC(\mathbb{Q})$ is a commutative ring and $SC_0(\mathbb{Q})$ is an ideal.

[2]For those who are familiar with rings and ideals: It is a general fact that if R is a commutative ring and I is an ideal, then the equivalence relation $\sim_I$ defined by letting $\langle s, t \rangle \in \sim_I \Longleftrightarrow s - t \in I$ is compatible with the addition, subtraction and multiplication operations on R, and therefore they give rise to operations of addition, subtraction and multiplication on the quotient $R/\sim_I$, so $R/\sim_I$ is a ring.

Definition.

i. A <u>real number</u> is a member of the quotient $SC(\mathbb{Q})/\sim$.

ii. The set of all real numbers is denoted by "$\mathbb{R}$", so $\mathbb{R} = SC(\mathbb{Q})/\sim$.

iii. If r is a rational number, then we can associate with r the "real number r", also called "r regarded as a real number". This number is denoted by $r_{\mathbb{R}}$, and is given by

$$r_{\mathbb{R}} = [r_{seq}]_{\sim} ,$$

where r_{seq} is the sequence given by

$$r_{seq} = \left\{ \langle n, r\rangle \,\middle|\, n \in \omega \right\}.$$

iv. If $a \in \mathbb{R}$,

iv.a. we say that a is <u>nonnegative</u>, and write $a \geq 0$, or $0 \leq a$, if $a = [r]_{\sim}$ for some sequence $r \in SC(\mathbb{Q})$ such that $r_n \geq 0$ for all $n \in \omega$.

iv.b. we say that a is <u>positive</u>, and write $a > 0$, or $0 < a$, if $a \geq 0$ and $a \neq 0_{\mathbb{R}}$.

v. If a, b are real numbers, we say that a is <u>smaller than or equal</u> to b if $b - a \geq 0$, and that a is <u>strictly smaller than</u> b if $b - a > 0$.

vi. An <u>upper bound</u> for a set S of real numbers a real number b such that $\forall x(x \in S \Longrightarrow x \leq b)$.

vii. A set S of real numbers is <u>bounded above</u> if S has an upper bound, i.e., if there exists a real number b such that $\forall x(x \in S \Longrightarrow x \leq b)$.

viii. A <u>least upper bound</u> for a set S of real numbers is a real number L such that

viii.a. L is an upper bound for S,

viii.b. If C is an arbitrary upper bound for S, then $C \geq L$. $\square$

Problem 2. ***Prove***, *using the construction of* $\mathbb{R}$ *given here*, that

1. "$\leq$" is a total ordering relation, that is:

1.a $\forall x(x \in \mathbb{R} \Longrightarrow x \leq x)$,

1.b $\forall x\, \forall y\Big((x \in \mathbb{R}\, \&\, y \in \mathbb{R}) \Longrightarrow (x \leq y \,.\mathrm{or}\, y \leq x)\Big)$, 1.c $\forall x \, \forall y \, \forall z \Big((x \in \mathbb{R} \& y \in \mathbb{R} \& z \in \mathbb{R} \& x \leq y \& y \leq z) \Longrightarrow (x \leq z) \Big)$,

1.d $\forall x \, \forall y \, \Big((x \in \mathbb{R} \,\&\, y \in \mathbb{R} \,\&\, x \leq y \,\& y \leq x) \Longrightarrow x = y \Big)$.

2. $\langle \mathbb{R}, \leq \rangle$ satisfies the ***completeness property***: *if S is a nonempty subset of $\mathbb{R}$ which is bounded above, then S has a least upper bound.* □

For problems 3 and 4, you are free to use the definition of "real number" given in the book and discussed in class, or the one presented earlier here. But I strongly recommend that you use the one given in the book, with Dedekind cuts.

Problem 3. ***Give a rigorous proof*** that, if we let

$$r = O_{\mathbf{R}} \cup \left\{ q \,\middle|\, q \in \mathbb{Q} \,\&\, q \cdot q < 2 \right\},$$

then r is a positive real number and $r \cdot r = 2_{\mathbf{R}}$. (Here: (a) the product of two nonnegative[3] real numbers is defined on Page 118 of the book; (b) "$2_{\mathbf{R}}$" is, of course, the real number $\left\{ q \,\middle|\, q \in \mathbb{Q} \,\&\, q < 2 \right\}$.) NOTE: At some point you will need to prove that if $q \in \mathbb{Q}$, $q > 0$, and $q < 2$, then you can write $q = q_1 q_2$, where $q_1 \in \mathbb{Q}$, $q_2 \in \mathbb{Q}$, $q_1 > 0$, $q_2 > 0$, $q_1 \cdot q_1 < 2$ and $q_2 \cdot q_2 < 2$. The most obvious choice would be to take $q_1 = q_2 = \sqrt{q}$, but this does not work because $\sqrt{q}$ could be irrational. To take care of this problem, you should pick q_1 "a little bit smaller than $\sqrt{q}$", and rational, and then let $q_2 = \frac{q}{q_1}$, so q_2 is "a little bit larger than $\sqrt{q}$", and rational, and $q_2 \cdot q_2$ is still < 2. To do this, I suggest you write $q = \frac{m}{n^2}$, with n very large and $m \in \omega$, and then let μ be the unique natural number such that $\mu^2 \leq m < (\mu + 1)^2$, and pick $q_1 = \frac{\mu}{n}$. □

Problem 4. For natural numbers k, the <u>k-th power</u> of a real number r is defined recursively by $r^0 = 1$, $r^{k+1} = r^k \cdot r$ for $k \in \omega$. ***Give a rigorous proof***, without using the completeness property, that if r is a nonnegative real number, $k \in \omega$, and $k > 0$, then $\exists s \in \mathbb{R} \, (s \geq 0 \,\&\, s^k = r)$. (That is, every nonnegative real number has a nonnegative k-th root.) HINT: Take $s = 0_{\mathbf{R}} \cup \{ \left\{ q \,\middle|\, q \in \mathbb{Q} \,\&\, q \geq 0 \,\&\, \exists t (t \in \mathbb{Q} \,\&\, t > q \,\&\, t^k \in r) \right\}$. □

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[3]Remember that "nonnegative" means "$ge0$", and "positive" means "> 0".