MATHEMATICS 361 — FALL 2019
SET THEORY
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SOME EXAMPLES OR PROOFS

You should use these proofs as examples of how you should write a proof. In particular, notice that ***a proof consists of clear, short, precise, steps, each one of which makes a definite assertion; and that at each point in the proof, it is clear (a) what we are assuming, (b) what we are trying to proof, (c) what are the values of all the letter variables.***

Example 1. ***Proof that $\emptyset$ is a function from $\emptyset$ to $\emptyset$.***

Proof. The sentence "f is a function from A to B" means "f is a function, $\mathrm{Dom}\, f = A$, and $\mathrm{Ran}\, f \subseteq B$." So we have to prove that

(I) $\emptyset$ is a function,
(II) $\mathrm{Dom}\,\emptyset = \emptyset$,
(III) $\mathrm{Ran}\,\emptyset \subseteq \emptyset$.

Proof that $\emptyset$ is a function. The sentence "f is a function" means "f is a relation, and whenever two ordered pairs $\langle u, v\rangle$, $\langle u, w\rangle$, it follows that $v = w$".

So we have to prove that

(Ia) $\emptyset$ is a relation, that is, $\emptyset$ is a set of ordered pairs, i.e.,

$$\forall x \Big(x \in \emptyset \Longrightarrow \exists u\, \exists v\, x = \langle u, v\rangle\Big) \tag{1}$$

("every member of $\emptyset$ is an ordered pair").

(Ib) whenever two ordered pairs $\langle u, v\rangle$, $\langle u, w\rangle$ belong to $\emptyset$, it follows that $v = w$. That is,

$$\forall u\, \forall v\, \forall w \Big(\big(\langle u, v\rangle \in \emptyset \,\&\, \langle u, w\rangle \in \emptyset\big) \Longrightarrow v = w\Big). \tag{2}$$

So we have to prove (Ia) and (Ib).

Proof of (Ia).

Let x be arbitrary.
We want to prove that

$$x \in \emptyset \Longrightarrow \exists u \, \exists v \, x = \langle u, v \rangle \,. \tag{3}$$

But (3) is an implication with a false premise (because $\forall x \, x \notin \emptyset$).

So (3) is true.

We have proved "$x \in \emptyset \Longrightarrow \exists u \, \exists v \, x = \langle u, v \rangle$" for arbitrary x.

Hence $\forall x \Big(x \in \emptyset \Longrightarrow \exists u \, \exists v \, x = \langle u, v \rangle \Big)$.

That is, $\emptyset$ is a relation, completing the proof of (Ia).

Proof of (Ib.) We have to prove that

$$\forall u \, \forall v \, \forall w \Big(\big(\langle u, v \rangle \in \emptyset \,\&\, \langle u, w \rangle \in \emptyset \big) \Longrightarrow v = w \Big) \,. \tag{4}$$

Let u, v, w be arbitrary.
We want to prove that

$$\big(\langle u, v \rangle \in \emptyset \,\&\, \langle u, w \rangle \in \emptyset \big) \Longrightarrow v = w \,. \tag{5}$$

But (5) is an implication with a false premise, (because the sentences "$\langle u, v \rangle \in \emptyset$" and "$\langle u, w \rangle \in \emptyset$" are false, since $\forall x \, x \notin \emptyset$).
Hence (5) is true.

We have proved (5) for arbitrary u, v, w.

Hence (4) is true, completing the proof of 1b.

Since we have proved (Ia) and (1b), we have proved (I), that is we have proved that $\emptyset$ is a function.

Proof of (II). We have to prove that $\mathrm{Dom}\, \emptyset = \emptyset$.

But $\mathrm{Dom}\, \emptyset = \{x | \exists y \langle x, y \rangle \in \emptyset$.

Let x be arbitrary.

Then the sentence "$\exists y \, \langle x, y \rangle \in \emptyset$ is false, because "$\langle x, y \rangle \in \emptyset$" is false for every y, since $\forall u \, u \notin \emptyset$.

So $x \notin \mathrm{Dom}\, \emptyset$.

Hence $\forall x \, x \notin \mathrm{Dom}\, \emptyset$.

This proves that $\operatorname{Dom}\emptyset = \emptyset$, completing the proof of (II).

Proof of (III). We have to prove that $\operatorname{Ran}\emptyset \subseteq \emptyset$.

Clearly, it suffices to prove that $\operatorname{Ran}\emptyset\emptyset$.

But $\operatorname{Ran}\emptyset = \{y|\exists x\langle x,y\rangle \in \emptyset$.

> Let y be arbitrary.
>
> Then the sentence "$\exists x\,\langle x,y\rangle \in \emptyset$ is false, because "$\langle x,y\rangle \in \emptyset$" is false for every x, since $\forall u\, u \notin \emptyset$.
>
> So $y \notin \operatorname{Ran}\emptyset$.

Hence $\forall y\, y \notin \operatorname{Ran}\emptyset$.

This proves that $\operatorname{Ran}\emptyset = \emptyset$, completing the proof of (III).

Since we have proved (I), (II), and (III), we have proved that $\boxed{\emptyset : \emptyset \mapsto \emptyset}$, that is, $\boxed{\emptyset \text{ is a function from } \emptyset \text{ to } \emptyset}$. **Q.E.D.**

Example 2. ***Prove that every natural number is a transitive set.***

Proof. The sentence "n is a natural number" means "n belongs to every inductive set". And the sentence "S is an inductive set" means "$\emptyset \in S \,\&\, \forall x\,(x \in S \Longrightarrow x \cup \{x\} \in S)$".

Hence "n is a natural number" means

$$\forall S\left(\Big(\emptyset \in S \,\&\, \forall x\,(x \in S \Longrightarrow x \cup \{x\} \in S)\Big) \Longrightarrow x \in S\right) \tag{6}$$

that is, "if S is an arbitrary inductive set, then $n \in S$").

The sentence "n is a transitive set" means "every member of a member of n is a member of n", that is

$$\forall m\,\forall q\Big((m \in n \,\&\, q \in m) \Longrightarrow q \in n\Big). \tag{7}$$

We want to prove that

$$\forall n\,(n \text{ is a natural number } \Longrightarrow n \text{ is a transitive set}). \tag{8}$$

We will prove this by induction. Let T be the set of all natural numbers n such that n is a transitive set.

We are going to prove that T is an inductive set.

For this purpose, we are going to prove that

(1) $0 \in T$,

(2) $\forall n\,(n \in T \Longrightarrow n \cup \{n\} \in T)$.

Proof of (1) (the basis step). We have to prove that 0 is a transitive set.

That is, we have to prove that

$$\forall m\,\forall q\Big((m \in \emptyset\,\&\,q \in m) \Longrightarrow q \in \emptyset\Big)\,. \tag{9}$$

Let m, q be arbitrary.

We want to prove that

$$(m \in \emptyset\,\&\,q \in m) \Longrightarrow q \in \emptyset\,. \tag{10}$$

Now, the sentence "$(m \in \emptyset, \&\,q \in m) \Longrightarrow q \in \emptyset$" is an implication whose premise is false (because "$m \in \emptyset$" is false, since $\forall x\, x \notin \emptyset$, and then the conjunction "$m \in \emptyset\,\&\,q \in m$" is false).

Hence the sentence "$(m \in \emptyset\,\&\,q \in m) \Longrightarrow q \in \emptyset$" is true.

So we have proved that (10) is true for arbitrary m, q.

Hence (9) is true.

Therefore the natural number 0 is a transitive set. Hence $\boxed{0 \in T}$, completing the proof of (1).

Proof of (2) (the inductive step). We have to prove that

$$\forall n\,(n \in T \Longrightarrow n \cup \{n\} \in T)\,. \tag{11}$$

Let n be arbitrary.

We want to prove that $n \in T \Longrightarrow n \cup \{n\} \in T$.

Assume $n \in T$.
We want to prove that $n \cup \{n\} \in T$.
Since $n \in T$, n is a natural number and a transitive set.
Then $n \cup \{n\}$ is a natural number.
We have to prove that $n \cup \{n\}$ is a transitive set.

That is, we have to prove that

$$\forall m\,\forall q\Big((m \in n \cup \{n\}\,\&\, q \in m) \Longrightarrow q \in n \cup \{n\}\Big)\,. \tag{12}$$

Let m, q be arbitrary.

We want to prove that

$$(m \in n \cup \{n\}\,\&\, q \in m) \Longrightarrow q \in n \cup \{n\}\,. \tag{13}$$

Assume that $m \in n \cup \{n\}\,\&\, q \in m$.

We want to prove that $q \in n \cup \{n\}$.

Clearly, $m \in n \cup \{n\}$.

So $m \in n$ or $m = n$.

If $m \in n$, then the facts that $q \in m$ and n is transitive imply that $q \in n$.

If $m = n$ then, obviously, the fact that $q \in m$ implies that $q \in n$.

So in both cases ($m \in n$ and $m = n$), it follows that $q \in n$.

And, since $n \subseteq n \cup \{n\}$, we can conclude that $q \in n \cup \{n\}$.

So we have proved that $q \in n \cup \{n\}$, assuming "$m \in n \cup \{n\}\,\&\, q \in m$".

Hence, we have proved (13).

Since we have proved (13) for arbitrary m, q, we have proved (12).

That is, we have proved that $n \cup \{n\}$ is a transitive set.

Since we have proved that $n \cup \{n\}$ is a transitive set, and we know that $n \cup \{n\}$ is a natural number, it follows that $n \cup \{n\} \in T$.

Since we have proved that $n \cup \{n\} \in T$ assuming that $n \in T$ we can conclude that

$$n \in T \Longrightarrow n \cup \{n\} \in T\,. \tag{14}$$

Since we have proved the implication (14) for arbitrary n, it follows that $\boxed{\forall n\,(n \in T \Longrightarrow n \cup \{n\} \in T)}$.

This completes the inductive step.

Since we have also carried out the basis step, by proving that $0 \in T$, it follows that $\boxed{\boxed{T \text{ is an inductive set}}}$.

We now complete our proof.

> Let n be an arbitrary natural number.
>
> Then n belongs to every inductive set.
>
> So in particular $n \in T$.
>
> But this says that n is a transitive set.

So we have finally proved our desired result, that is, that

$\boxed{\text{every natural number is a transitive set}}$. **Q.E.D.**

Example 3. ***Proof that if*** n ***is a natural number then every member of*** n ***is a natural number.***

Let ω be the set of all natural numbers. That is, let

$$\omega = \{\, n | n \text{ is a natural number } \}, \tag{15}$$

or, equivalently,

$$\omega = \left\{ n \,\middle|\, \forall S \left(\left(\emptyset \in S \,\&\, \forall x \left(x \in S \Longrightarrow x \cup \{x\} \in S \right) \right) \Longrightarrow n \in S \right) \right\}. \tag{16}$$

We want to prove that if $n \in \omega$ and $m \in n$ then $m \in \omega$. (That is, we want to prove that $\forall n \left(n \in \omega \Longrightarrow n \subseteq \omega \right)$. Equivalently, we want to prove that ω is a transtive set.)

We are going to prove this by induction.

Let T be the set of all natural numbers n such that $n \subseteq \omega$.

We are going to prove that T is inductive.

For this purpose, we are going to prove that

(1) $0 \in T$,

(2) $\forall n \, (n \in T \Longrightarrow n \cup \{n\} \in T)$.

Proof of (1) (the basis step). We have to prove that $0 \subseteq \omega$.

But we know that 0 is a subset of every set, because $0 = \emptyset$, and $\forall x \, \emptyset \subseteq x$.

Hence $0 \subseteq \omega$.

Therefore $\boxed{0 \in T}$, completing the proof of (1).

Proof of (2) (the inductive step). We have to prove that

$$\forall n \, (n \in T \Longrightarrow n \cup \{n\} \in T) \, . \tag{17}$$

Let n be arbitrary.

Assume $n \in T$.
We want to prove that $n \cup \{n\} \in T$.
Since $n \in T$, it follows that $n \in \omega$ and $n \subseteq \omega$.
Then $n \cup \{n\}$ is a natural number.
We have to prove that $n \cup \{n\} \subseteq \omega$.
For this purpose, it suffices to prove that $n \subseteq \omega$ and $\{n\} \subseteq \omega$.
But we already know that $n \subseteq \omega$.
As for $\{n\}$, to prove that $\{n\} \subseteq \omega$ we have to prove that $n \in \omega$.
And, indeed, we know that $n \in \omega$, so $\{n\} \subseteq \omega$.
So we have proved that $n \subseteq \omega$ and $\{n\} \subseteq \omega$.
It follows that $n \cup \{n\} \subseteq \omega$.

Since we have proved that $n \cup \{n\} \in T$ assuming that $n \in T$ we can conclude that

$$n \in T \Longrightarrow n \cup \{n\} \in T \, . \tag{18}$$

Since we have proved the implication (18) for arbitrary n, it follows that $\boxed{\forall n \, (n \in T \Longrightarrow n \cup \{n\} \in T)}$.

This completes the inductive step.

Since we have also carried out the basis step, by proving that $0 \in T$, it follows that $\boxed{\boxed{T \text{ is an inductive set}}}$.

We now complete our proof.

Let n be an arbitrary natural number.

Then n belongs to every inductive set.

So in particular $n \in T$.

But this says that $n \subseteq \omega$.

So we have finally proved our desired result, that is, that every natural number is a subset of ω, i.e., that

$\boxed{\text{every natural number is a set of natural numbers}}$. **Q.E.D.**