Problem 1. As explained in the Wikipedia article on the Lotka-Volterra equations, the function $K$ given by

$$K(x, y) = y^\alpha e^{-\beta y}x^\gamma e^{-\delta x}$$

is a constant of motion. (NOTES: We are only interested in the nonnegative quadrant, i.e., the points $(x, y)$ such that $x \geq 0$ and $y \geq 0$. This is because negative numbers of rabbits or wolves do not make sense. Also: in the Wikipedia article, $x$ is the number of prey (for example, rabbits), that we called $r$ in class, and $y$ is the number of predators (for example, wolves), that we called $w$ in class.)

1. **Compute** the derivatives $\frac{\partial K}{\partial x}$ and $\frac{\partial K}{\partial y}$. (I strongly recommend that you express these two derivatives in the form of some function times $K$. The functions that you get this way are very simple, and it’s very easy to tell when they are positive and when they are negative.)

2. **Verify** that $K$ is a constant of motion, by computing the derivative of $K$ along solutions of the system of differential equations and showing that this derivative is zero. (You will have to use the Chain Rule for derivatives of functions of several variables that you learned in one of your Calculus courses).

3. **Verify** that

   a. There is a vertical line $L_{vert}$ such that $\frac{\partial K}{\partial x} > 0$ to the left of that line, and $\frac{\partial K}{\partial x} < 0$ to the right. (This means that, if you move from left to right, the value of $K$ increases up to a certain point and then decreases.)

   b. There is a horizontal line $L_{hor}$ such that $\frac{\partial K}{\partial y} < 0$ above that line, and $\frac{\partial K}{\partial y} > 0$ below. (This means that, if you move vertically up, the value of $K$ increases up to a certain point and then decreases.)
c. The two lines \( L_{\text{vert}} \) and \( L_{\text{hor}} \) meet precisely at the equilibrium point.

d. If you start at time zero at the equilibrium point \((x_{eq}, y_{eq})\) and move in any direction along a straight line (i.e., if you move so that your position \((x(t), y(t))\) at time \(t\) is given by

\[
    x(t) = x_{eq} + t \cos \theta \quad \text{and} \quad y(t) = y_{eq} + t \sin \theta, \quad (0.1)
\]

where \(\theta\) is some angle), then \(K\) decreases.

4. **Deduce** from the previous results that the function \(K\) has a maximum at the equilibrium point. (Please do **not** do this by using the second derivatives criterion. That would be bad for two reasons: first, you do not want to have to do extra work computing the second derivatives, and, second, the second derivatives criterion only gives you a sufficient condition for a point to be a **local** maximum, and here we are looking for a **global** maximum.)

5. **Conclude** from the previous results that the level curves (i.e., the curves \(\lambda_c\) given by the equation \(K(x, y) = c\), for all numbers \(c\) smaller than the maximum value \(K_{\text{max}}\) of \(K\)) look like circles (not exactly circles, but similar) going around the equilibrium point. (NOTE: To make this completely precise and rigorous would require more sophisticated mathematics than you are expected to know for this course. Just give a plausibility argument, using pictures if you want. But if you want to give it a try, you could try to prove rigorously that, if \(c < K_{\text{max}}\), and you move in any direction \(\theta\) following \((0.1)\), then there will be exactly one time \(t(\theta)\) when the value of \(K\) will become \(c\), and this will give you the equation of the level curve \(\lambda_c\) in polar coordinates with center at the equilibrium point.)

6. **Conclude** from the previous results that the solutions of the Lotka-Volterra differential equations are periodic functions of time.