MATHEMATICS 501 — FALL 2016

Theory of functions of a real variable I

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HOMEWORK ASSIGNMENT NO. 1, DUE ON FRIDAY, SEPTEMBER 16

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Some of these problems may be hard. Do not worry if you cannot do everything. Just do what you can.

Problem 1. ***Prove*** that if (X,ρ) is a metric space, S is a subset of X, and S satisfies the following conditions:

(*1) *For every positive*[1] *real number* ε *there exists a compact subset* K_ε *of* X *such that*

$$\sup\Big\{\inf\{\rho(x,y) : y\in K_\varepsilon\} : x\in S\Big\}\leq\varepsilon\,.$$

(*2) S is complete.

then S is compact.

Problem 2. ***Construct*** a bijective map f from the real line $\mathbb{R}$ onto the closed interval $[0,1]$. (NOTE: The existence of such a map follows easily using the Schröder-Bernstein Theorem. Here you are asked to construct a map explicitly, without using the Schröder-Bernstein Theorem.)

Problem 3. A subset I of the extended real line $\bar{\mathbb{R}}$ is an <u>interval</u> if it has the following property:

(#) *Whenever* a,b,c *are members of* $\bar{\mathbb{R}}$ *such that* $a<b<c$, $a\in I$, *and* $c\in I$, *it follows that* $b\in I$.

Prove the following sstatements:

1. The empty set is an interval.

[1] Throughout this course, "positive" meeans ">0", and "nonnegative" means "≥ 0".

2. If $a \in \bar{\mathbb{R}}$, then the set $\{a\}$ is an interval.

3. If $I \subseteq \bar{\mathbb{R}}$, then I is an interval if and only if there exist $a, b \in \bar{\mathbb{R}}$ such that I is one of the following four sets:

$$\{x \in \bar{\mathbb{R}} : a < x < b\}, \tag{0.1}$$
$$\{x \in \bar{\mathbb{R}} : a \leq x \leq b\},, \tag{0.2}$$
$$\{x \in \bar{\mathbb{R}} : a < x \leq b\}, \tag{0.3}$$
$$\{x \in \bar{\mathbb{R}} : a \leq x < b\}. \tag{0.4}$$

4. If we use exactly the same definition of "interval" for subsets of the extended rational line $\bar{\mathbb{Q}}$, then the result of Part 3 is not true.

   NOTE: The extended rational line is the set $\bar{\mathbb{Q}}$ given by $\bar{\mathbb{Q}} = \mathbb{Q} \cup \{-\infty, \infty\}$. An interval in $\bar{\mathbb{Q}}$ is a subset I of $\bar{\mathbb{Q}}$ such that, whenever a, b, c are members of $\bar{\mathbb{Q}}$ such that $a < b < c$, $a \in I$, and $c \in I$, it follows that $b \in I$. The "statement of part 3" is the statement that

   (#) If $I \subseteq \bar{\mathbb{Q}}$, then I is an interval if and only if there exist $a, b \in \bar{\mathbb{Q}}$ such that I is one of the following four sets:

$$\{x \in \bar{\mathbb{Q}} : a < x < b\}, \tag{0.5}$$
$$\{x \in \bar{\mathbb{Q}} : a \leq x \leq b\},, \tag{0.6}$$
$$\{x \in \bar{\mathbb{Q}} : a < x \leq b\}, \tag{0.7}$$
$$\{x \in \bar{\mathbb{Q}} : a \leq x < b\}. \tag{0.8}$$

Problem 5. ***Prove*** that if A is an infinite set then A has a partition consisting of countably infinite sets. (That is: there exists a set $\mathcal{P}$ such that: (1) every member of $\mathcal{P}$ is a countably infinite subset of A, (2) if X, Y are any two members of $\mathcal{P}$, then either $X = Y$ or $X \cap Y = \emptyset$, and (3) $\bigcup \mathcal{P} = A$. Recall that a set X is finite if there exists a bijection from X onto the set $\mathbb{N}_n$ for some nonnegative integer n, where $\mathbb{N}_n = \{k \in \mathbb{N} : k \leq n\}$, so $\mathbb{N}_0 = \emptyset$, $\mathbb{N}_1 = \{1\}$, $\mathbb{N}_2 = \{1, 2\}$, and so on. A set is infinite if it is not finite. A set X is countably infinite if there exists a bijection from X onto $\mathbb{N}$.) HINT: For this proof it is essential that you use the Axiom of Choice or something equivalent, such as Zorn's lemma or the Hausdorff maximal principle.