Problem 1. Prove that if \((X, \rho)\) is a metric space, \(S\) is a subset of \(X\), and \(S\) satisfies the following conditions:

\[ (*1) \text{ For every positive real number } \varepsilon \text{ there exists a compact subset } K_\varepsilon \text{ of } X \text{ such that } \sup \left\{ \inf \{ \rho(x, y) : y \in K_\varepsilon \} : x \in S \right\} \leq \varepsilon. \]

\[ (*2) \text{ S is complete.} \]

then \( S \) is compact.

Problem 2. Construct a bijective map \( f \) from the real line \( \mathbb{R} \) onto the closed interval \([0, 1]\). (NOTE: The existence of such a map follows easily using the Schröder-Bernstein Theorem. Here you are asked to construct a map explicitly, without using the Schröder-Bernstein Theorem.)

Problem 3. A subset \( I \) of the extended real line \( \bar{\mathbb{R}} \) is an interval if it has the following property:

\[ (#) \text{ Whenever } a, b, c \text{ are members of } \bar{\mathbb{R}} \text{ such that } a < b < c, a \in I, \text{ and } c \in I, \text{ it follows that } b \in I. \]

Prove the following statements:

1. The empty set is an interval.

\(^1\text{Throughout this course, “positive” means “}> 0”, and “nonnegative” means “} \geq 0”.\)
2. If \( a \in \bar{\mathbb{R}} \), then the set \( \{a\} \) is an interval.

3. If \( I \subseteq \bar{\mathbb{R}} \), then \( I \) is an interval if and only if there exist \( a, b \in \bar{\mathbb{R}} \) such that \( I \) is one of the following four sets:
   \[
   \begin{align*}
   \{x \in \bar{\mathbb{R}} : a < x < b\}, & \quad (0.1) \\
   \{x \in \bar{\mathbb{R}} : a \leq x \leq b\}, & \quad (0.2) \\
   \{x \in \bar{\mathbb{R}} : a < x \leq b\}, & \quad (0.3) \\
   \{x \in \bar{\mathbb{R}} : a \leq x < b\}. & \quad (0.4)
   \end{align*}
   \]

4. If we use exactly the same definition of "interval" for subsets of the extended rational line \( \bar{\mathbb{Q}} \), then the result of Part 3 is not true.

**NOTE:** The extended rational line is the set \( \bar{\mathbb{Q}} \) given by \( \bar{\mathbb{Q}} = \mathbb{Q} \cup \{-\infty, \infty\} \). An interval in \( \bar{\mathbb{Q}} \) is a subset \( I \) of \( \bar{\mathbb{Q}} \) such that, whenever \( a, b, c \) are members of \( \bar{\mathbb{Q}} \) such that \( a < b < c \), \( a \in I \), and \( c \in I \), it follows that \( b \in I \). The "statement of part 3" is the statement that

(\#) If \( I \subseteq \bar{\mathbb{Q}} \), then \( I \) is an interval if and only if there exist \( a, b \in \bar{\mathbb{Q}} \) such that \( I \) is one of the following four sets:
   \[
   \begin{align*}
   \{x \in \bar{\mathbb{Q}} : a < x < b\}, & \quad (0.5) \\
   \{x \in \bar{\mathbb{Q}} : a \leq x \leq b\}, & \quad (0.6) \\
   \{x \in \bar{\mathbb{Q}} : a < x \leq b\}, & \quad (0.7) \\
   \{x \in \bar{\mathbb{Q}} : a \leq x < b\}. & \quad (0.8)
   \end{align*}
   \]

**Problem 5.** Prove that if \( A \) is an infinite set then \( A \) has a partition consisting of countably infinite sets. (That is: there exists a set \( \mathcal{P} \) such that:
   (1) every member of \( \mathcal{P} \) is a countably infinite subset of \( A \), (2) if \( X, Y \) are any two members of \( \mathcal{P} \), then either \( X = Y \) or \( X \cap Y = \emptyset \), and (3) \( \bigcup \mathcal{P} = A \).

Recall that a set \( X \) is finite if there exists a bijection from \( X \) onto the set \( \mathbb{N}_n \) for some nonnegative integer \( n \), where \( \mathbb{N}_n = \{k \in \mathbb{N} : k \leq n\} \), so \( \mathbb{N}_0 = \emptyset \), \( \mathbb{N}_1 = \{1\}, \mathbb{N}_2 = \{1, 2\} \), and so on. A set is infinite if it is not finite. A set \( X \) is countably infinite if there exists a bijection from \( X \) onto \( \mathbb{N} \).

HINT: For this proof it is essential that you use the Axiom of Choice or something equivalent, such as Zorn’s lemma or the Hausdorff maximal principle.