MATHEMATICS 501 — FALL 2016

Theory of functions of a real variable I

H. J. Sussmann

HOMEWORK ASSIGNMENT NO. 2, DUE ON FRIDAY, SEPTEMBER 23

Problem 1.

1. Let $\mathcal{B}$ be the set of all subsets S of $\mathbb{R}$ such that S is a finite union of intervals. (Recall that an interval is a subset I of $\mathbb{R}$ such that whenever $a, b, c \in \mathbb{R}$, $a < b < c$, and a and c belong to $\mathbb{R}$, it follows that $b \in I$.) ***Prove*** that $\mathcal{B}$ is an algebra of subsets of $\mathbb{R}$.

2. Let $\mathcal{C}$ be the set of all subsets S of $\mathbb{R}$ such that S is a finite or countably infinite union of intervals. Is $\mathcal{C}$ an algebra of subsets of $\mathbb{R}$?

In the following problems, we deal with Riemann integrals. The definitions used here of "Riemann integral" and "Riemann integrability" are as follows:

If $a, b \in \mathbb{R}$, $a \leq b$, and $f : [a, b] \mapsto \mathbb{R}$ is a function, then

1. A partition of $[a, b]$ is a finite sequence $\pi = (t_0, t_1, \ldots, t_n)$ of real numbers such that $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$.

2. A sequence of sampling points for a partition $\pi = (t_0, t_1, \ldots, t_n)$ of $[a, b]$ is a finite sequence $\sigma = (s_1, s_2, \ldots, s_n)$ of real numbers that satisfies $t_{j-1} \leq s_j \leq t_j$ for $j = 1, \ldots, n$.

3. If $\delta \in \mathbb{R}$ and $\delta > 0$, then a partition $\pi = (t_0, t_1, \ldots, t_n)$ of $[a, b]$ is δ-fine if $t_j - t_{j-1} \leq \delta$ for every $j \in \{1, \ldots, n\}$.

4. If $\pi = (t_0, t_1, \ldots, t_n)$ is a partition of $[a, b]$, and $\sigma = (s_1, s_2, \ldots, s_n)$ is a sequence of sampling points for π then the Riemann sum $S(f, \pi, \sigma)$ is the number given by

$$S(f, \pi, \sigma) = \sum_{j=1}^{n} f(s_j)(t_j - t_{j-1}) .$$

5. A real number I is said to be the Riemann integral $\int_a^b f(x)dx$ if

(#) For every positive real number ε there exists a positive real number δ such that $|I - S(f,\pi,\sigma)| < \varepsilon$ whenever $\pi = (t_0, t_1, \ldots, t_n)$ is a δ-fine partition of $[a,b]$ and σ is a sequence of sampling points for π.

It is is easy to see that if a number I such that (#) holds exists, then it is unique. The function f is Riemann integrable if there exists I such that (#) holds.)

Problem 2. ***Prove*** that if $a, b \in \mathbb{R}$, $a < b$, and $f : [a,b] \mapsto \mathbb{R}$ is a Riemman integrable function, then f is bounded.

Problem 3. Let (X, ρ) be a metric space. Define a set $\hat{X}_0$ as follows: $\hat{X}_0$ is the set of all Cauchy sequences of members of X. (Recall that a Cauchy sequence of members of a metric space (A, d) is a sequence $\{a_n\}_{n=1}^{\infty}$ of members of A such that for every positive real number ε there exists a positive integer N such that $d(x_n, x_m) < \varepsilon$ whenever n, m are positive integers such that $n \geq N$ and $m \geq N$.) Then define a binary relation $\sim$ on $\hat{X}_0$ by declaring two Cauchy sequences $\mathbf{x} = \{x_n\}_{n=1}^{\infty}$, $\mathbf{y} = \{y_n\}_{n=1}^{\infty}$, to be equivalent (and writing "$\mathbf{x} \sim \mathbf{y}$") if $\lim_{n\to\infty} \rho(x_n, y_n) = 0$. Let $\hat{X}$ be the quotient of $\hat{X}_0$ by the equivalence relation $\sim$. (If $\mathbf{x} \in \hat{X}_0$, then

1. The equivalence class of $\mathbf{x}$ is the set of all $\mathbf{y} \in \hat{X}_0$ such that $\mathbf{y} \sim \mathbf{x}$.
2. We write "$[\mathbf{x}]$" to denote the equivalence class of $\mathbf{x}$.

The set $\hat{X}$ is the set of all equivalence classes, i.e., the set of all sets C such that $C = [\mathbf{x}]$ for some $\mathbf{x} \in \hat{X}_0$. Then every member C of $\hat{X}$ is $[\mathbf{x}]$ for some $\mathbf{x} \in \hat{X}_0$, and two members $[\mathbf{x}]$, $[\mathbf{y}]$ of $\hat{X}$ are equal if and only if $\mathbf{x} \sim \mathbf{y}$.)

I. ***Prove*** that, if $C = [\mathbf{x}]$ and $D = [\mathbf{y}]$ are two members of the set $\hat{X}$, $\mathbf{x} = \{x_n\}_{n=1}^{\infty}$ and $\mathbf{y} = \{y_n\}_{n=1}^{\infty}$, then

   a. The limit $\lambda = \lim_{n\to\infty} \rho(x_n, y_n)$ exists,

   b. λ is a nonnegative real number,

   c. $\lambda = 0$ if and only if $\mathbf{x} \sim \mathbf{y}$.

   d. λ does not depend on how the representatives $\mathbf{x}$, $\mathbf{y}$ of the classes C, D are chosen. (That is: If $\mathbf{x}' = \{x'_n\}_{n=1}^{\infty}$ is another member of C, and $\mathbf{y}' = \{y'_n\}_{n=1}^{\infty}$ is another member of D, then it follows that $\lim_{n\to\infty} \rho(x_n, y_n) = \lim_{n\to\infty} \rho(x'_n, y'_n)$.)

II. It follows from the results of Part I that, if C, D are any two members of $\hat{X}$, the number λ of Part I.a is well defined. Let us use $\hat{\rho}(C, D)$ to denote this number. ***Prove*** that $(\hat{X}, \hat{\rho})$ is a complete metric space. (That is, you must prove first that the function $\hat{\rho} : \hat{X} \times \hat{X} \mapsto \mathbb{R}$ is a metric, and then you have to show that the metric space $(\hat{X}, \hat{\rho})$ is complete.)

III. For each $x \in X$, let $\check{x}$ be the sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n = x$ for every n. Define a map $\iota : X \mapsto \hat{X}$ by letting $\iota(x) = [\hat{x}]$. **Prove** that

α. The map ι is injective.

β. $\hat{\rho}(\iota(x), \iota(y)) = \rho(x, y)$ for all $x, y \in X$.

γ. The image $\iota(X)$ of the map ι is dense in $\hat{X}$.

NOTES:

A. The complete metric space $(\hat{X}, \hat{\rho})$ constructed in Problem 2 is called the <u>completion</u> of (X, ρ).

B. If we identify X with $\iota(X)$, we have shown that

The metric space (X, ρ) is a dense subspace of the complete metric space $(\hat{X}, \hat{\rho})$.

(If you find the idea of "identifying" the sets X and $\iota(X)$ confusing, just think of it this way: $\hat{X}$ consists of all the classes $[\mathbf{x}]$, for all $\mathbf{x} \in \hat{X}_0$. Some of these classes are $\iota(x)$ for some $x \in X$, and other classes do not arise in this way. Whenever a member C of $\hat{X}$ actually is a class $\iota(x)$, for an $x \in X$, remove C from $\hat{X}$ and put x instead. In this way, the new set $\hat{X}$ consists of all the members of X, together with all the classes C that are not $\iota(x)$ for any $x \in X$. So we have, quite literally, removed $\iota(X)$ from $\hat{X}$, and put X instead.

Problem 4. Let (X, ρ), (Y, σ) be metric spaces. A mapping $f : X \mapsto Y$ is <u>Lipschitz</u> if there exists a nonnegative real number c such that

$$\sigma(f(x), f(x')) \leq c\rho(x, x') \qquad \text{for all} \quad x, x' \in X\,. \tag{0.1}$$

(A nonnegative real number c that satisfies condition (0.1) is said to be a <u>Lipschitz constant</u> for f.)

Prove that if (X, ρ), (Y, σ) are metric spaces, $f : X \mapsto Y$ is Lipschitz, (Y, σ) is complete, and $(\hat{X}, \hat{\rho})$ is the completion of (X, ρ), then there exists a unique map $\hat{f} : \hat{X} \mapsto Y$ such that

1. $\hat{f}$ is continuous,
2. $\hat{f}(x) = f(x)$ whenever $x \in X$.

Problem 5. Fix real numbers a, b such that $a < b$. Let X be the space of all continuous functions $f : [a, b] \mapsto \mathbb{R}$. (This is the set that was called $C^0([a, b], \mathbb{R})$ in class.) Let $I : X \mapsto \mathbb{R}$ be the mapping defined by

$$I(f) = \int_a^b f(x)dx\,. \tag{0.2}$$

Let $\rho_{L^1} : X \times X \mapsto \mathbb{R}$ be the function defined by

$$\rho_{L^1}(f, g) = \int_a^b |f(x) - g(x)|dx\,. \tag{0.3}$$

1. ***Prove*** that ρ_{L^1} is a metric on X, so (X, ρ_{L^1}) is a metric space.
2. ***Prove*** that I, regarded as a map from (X, ρ_{L^1}) to $\mathbb{R}$, is Lipschitz, and determine the Lipschitz constant c.
3. ***Conclude*** from the above that there exists a unique way to extend the map $I : X \mapsto \mathbb{R}$ given by $I(f) = \int_a^b f(x)dx$ for $f \in X$ to a map $\hat{I}$ which is defined on the (possibly) much larger set $\hat{X}$ and continuous from $(\hat{X}, \hat{\rho})$ to $\mathbb{R}$.

NOTE: Having done Problem 6, you know in principle, that $I(f)$ can be defined by the procedure specified there for objects f in a much larger set than the set of continuous functions. But we still do not have any concrete knowledge of what those objets are. They are just equivalence classes of Cauchy sequences of continuous functions. It would be nicer if we could think of the members of $\hat{X}$ as true functions. If we could do that, then we would have extended the notion of "integral" to a much larger set of functions.

The following problem discusses a very simple example where that can be done.

Problem 7. Let us now work on the interval $[0,1]$. Define $f_n : [0,1] \mapsto \mathbb{R}$, for $n \in \mathbb{N}$, by letting

$$f_n(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if} \quad \frac{1}{n} \leq x \leq 1 \\ \sqrt{n} & \text{if} \quad 0 \leq x \leq \frac{1}{n}\,. \end{cases} \tag{0.4}$$

I. ***Prove*** that

1. The functions f_n are continuous on $[0,1]$ for each n.
2. The sequence $\mathbf{f} = \{f_n\}_{n=1}^{\infty}$ is Cauchy with respect to the metric ρ_{L^1}.

II. ***Conclude*** from Part I that the limit $f = \lim_{n\to\infty} f_n$ exists as a member of the completion $\hat{X}$ of X with respect to the metric ρ_{L^1}.

IV. ***Compute*** the integrals $I(f_n) = \int_0^1 f_n(x)dx$, and then compute the limit $\theta = \lim_{n\to\infty} I(f_n)$.

V. Let g be the function from $[0,1]$ to $\mathbb{R}$ given by

$$g(x) = \frac{1}{\sqrt{x}} \qquad \text{if} \quad x > 0\,,$$

and supplement this by defining $g(0)$ to be equal to ξ, where ξ is a real number that you are free to choose as you wish.

Observe[1] that

1. The functions f_n "converge to g" in the sense that

$$\lim_{n\to\infty} f_n(x) = g(x) \qquad \text{for every} \quad x \quad \text{such that} \quad x > 0\,.$$

2. There is no way to choose the number ξ so that

$$\lim_{n\to\infty} f_n(x) = g(x) \qquad \text{for every} \quad x \in [0,1]\,. \tag{0.5}$$

3. On the other hand, if you allow "functions with infinite values" (i.e., functions from $[0,1]$ to the extended real line $\bar{\mathbb{R}}$), then you can choose ξ to be $+\infty$, and (0.5) will be true.

[1] When I ask you to "observe" or "conclude" something, you don't have to write anything. I am just putting this in in order to make sure that you draw the conclusion that I want you to draw, i.e., that you get my point.

VI. ***Conclude*** from the above, that the object f defined in Part II above, which is in principle an equivalence class of Cauchy sequences[2], "can be regarded" (whatever that means) as a true function, namely, the function g defined above. But ***observe*** that this function is not a true function defined in a unique way. The value of g at 0 does not matter, really, and there is no natural way to choose it.

VII. ***Prove*** rigorously that the function g defined above (with ξ chosen in any way you like) is ***not*** Riemann integrable.

NOTE: For this problem, the function g could still be said to be "integrable" using a small extension of the notion of Riemann integrability. For example, we could consider "improper Riemann integrals": we could say that a function $h : [0,1] \mapsto \mathbb{R}$ has an improper integral if: (a) the restriction of f to the interval $[\varepsilon, 1]$ is Riemann integrable for every $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < 1$, and (b) the limit $\lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 h(x)dx$ exists.

With this notion of integrability, our function g is integrable. In next week's assignment, we will see an example where things are not so simple: we will construct a member f of $\bar{X}$, and a function $g : [0,1] \mapsto \mathbb{R}$ that ought to be "f regarded as a function", but is such that the integral of g from 0 to 1 does not exist as an improper integral.

[2] Precisely, f is the set of all sequences $\{h_n\}_{n=1}^\infty$ of continuous functions on $[0,1]$ that satisfy $\lim_{n\to\infty} \int_0^1 |h_n(x) - f_n(x)|dx = 0$.