MATHEMATICS 501 — FALL 2016

Theory of functions of a real variable I

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HOMEWORK ASSIGNMENT NO. 3, DUE ON FRIDAY, OCTOBER 7

Outer measure and null sets. If $S \subset \mathbb{R}$, the Lebesgue outer measure of S is the (possibly extended) real number $\mu^*(S)$ defined by

$$\mu^*(S) = \inf\left\{ \sum_{j=1}^{\infty} |I_j| : \{I_j\}_{j=1}^{\infty} \in \mathcal{OIC}(S) \right\},$$

where

1. $\mathcal{OI}$ is the set of all sequences $\{I_j\}_{j=1}^{\infty}$ of bounded open intervals of $\mathbb{R}$,
2. If $I = (a,b)$ is a bounded open interval of $\mathbb{R}$, the length of I is the number $|I|$ given by
$$|I| = b - a\,.$$
3. If $\mathbf{I} = \{I_j\}_{j=1}^{\infty} \in \mathcal{OI}$, and $S \subset \mathbb{R}$, we say that $\mathbf{I}$ covers S if
$$S \subset \bigcup_{n=1}^{\infty} I_n\,.$$
4. We use $\mathcal{OIC}(S)$ to denote the set of all sequences $\mathbf{I} \in \mathcal{OI}$ such that $\mathbf{I}$ covers S.

A null subset of $\mathbb{R}$ is a subset S of $\mathbb{R}$ such that $\mu^*(S) = 0$.

A property $P(x)$ of real numbers x holds almost everywhere on some set S if it holds for all $x \in S$ except possibly for the values of x belonging to a null set. (Precisely $P(x)$ holds almost everywhere if there exists a null set N such that $P(x)$ holds for all $x \in S\backslash N$.) For example, "x is irrational" holds almost everywhere (or, alternatively, "almost every real number is irrational"), because "x is irrational" is true except when x is rational, i.e., when $x \in \mathbb{Q}$, and the set $\mathbb{Q}$ is countable and hence a null set.

Problem 1. Let $\mathcal{N}$ be the set of all null subsets of $\mathbb{R}$. ***Prove*** that

$$\mathrm{card}(\mathcal{N}) > \mathrm{card}(\mathbb{R}).$$

(This may require finding first a null set S such that $\mathrm{card}(S) = \mathrm{card}(\mathbb{R})$.)

Problem 2. Let $\bar{\mathcal{N}}$ be the set of all subsets of $\mathbb{R}$ such that either (a) S is a null set, or (b) $\mathbb{R}\backslash S$ is a null set. ***Prove*** that $\bar{\mathcal{N}}$ is a σ-algebra of subsets of $\mathbb{R}$ and the restriction of μ^* to $\bar{\mathcal{N}}$ is a non-semifinite measure on $\mathcal{N}$.

Problem 3. ***Prove*** that if $a, b \in \mathbb{R}$, $a < b$. and $f : [a, b] \mapsto \mathbb{R}$ is a function, then a necessary condition for f to be Riemann integrable is that the set $DISC(f)$ of points of discontinuity of f be a null set. (By definition, a point of discontinuity of f is a point $x \in [a, b]$ such that

(a) there exists a real number ε such that $\varepsilon > 0$ and

(a1) there exists a sequence $\mathbf{x} = \{x_n\}_{n=1}^{\infty}$ of members of $[a, b]$ such that $\lim_{n\to\infty} x_n = x$, and $|f(x_n) - f(x)| \geq \varepsilon$ for every n.

For any given ε, a point of ε-discontinuity is a point x such that (a1) holds. So x is a point of discontinuity if and only if it is a point of ε-discontinuity for some ε such that $\varepsilon > 0$.) (HINT: Prove that for each positive ε the set of points of ε-discontinuity must be a null set.)

Problem 4. Let us work on the interval $[0, 1]$. We are going to construct a function $f : [0, 1] \mapsto \mathbb{R}$ that has a lot of singularities.

1. Start with an enumeration $\{r_m\}_{m=1}^{\infty}$ of the rational numbers that belong to $[0, 1]$. (That is, $\{r_m\}_{m=1}^{\infty}$ is a sequence such that the map $\mathbb{N} \ni m \mapsto r_m \in [0, 1]$ is a bijection from $\mathbb{N}$ onto $\mathbb{Q} \cap [0, 1]$.)

2. Define
$$f(x) = \sum_{m=1}^{\infty} \frac{2^{-m}}{\sqrt{|x - r_m|}}. \tag{0.1}$$
(NOTE: For each x, the summands of the right-hand side of (0.1) are nonnegative extended[1] real numbers. Hence the sum exists, as an extended real number, and is equal to the supremum of the partial sums

[1] I say "extended" because, for the function $x \mapsto \frac{2^{-m}}{\sqrt{|x-r_m|}}$, the value when $x = r_m$ is $+\infty$.

$S_N(x)$ given by

$$S_N(x) = \sum_{m=1}^{N} \frac{2^{-m}}{\sqrt{|x - r_m|}}\,.$$

If we let $g_m(x) = \frac{1}{\sqrt{|x-r_m|}}$, then the function g_m is exactly the same as the function g of Problem 7 of Homework 2 —i.e., the function $x \mapsto \frac{1}{\sqrt{|x|}}$—except that the singularity has been moved from 0 to r_m. What are doing is summing all these functions, for all r_m, in order to produce a function that has singularities at all the points of $\mathbb{Q} \cap [0,1]$. The factors 2^{-m} are put in so as to make the series convergent.)

Prove that $f(x)$ is finite almost everywhere on $[0,1]$.

Problem 5. *Prove* the following statement:

(*) Let d be a natural number. Let $R, R_1, \dots, R_n$ be rectangles in $\mathbb{R}^d$ such that

$$R \subset R_1 \cup R_2 \cup \cdots \cup R_n\,.$$

Then

$$|R| \le |R_1| + |R_2| + \cdots + |R_n|\,. \tag{0.2}$$

NOTE: The relevant definitions here are as follows:

1. A <u>closed rectangle</u> in $\mathbb{R}^d$ is a set R of the form

$$R = \prod_{j=1}^{d} [a_j, b_j]\,, \tag{0.3}$$

where $a_1, b_1, a_2, b_2, \dots, a_d, b_d$ are real numbers. (Notice that if $a_j > b_j$ for some j, then R is empty. So a closed rectangle is either the empty set or a set R of the form (0.3) with $a_j \le b_j$ for $j = 1, 2, \dots, d$.) In particular, a closed rectangle in $\mathbb{R}^d$ is a compact subset of $\mathbb{R}^d$.

2. A <u>rectangle</u> in $\mathbb{R}^d$ is a set S such that

$$\mathrm{Int}(R) \subset S \subset R \tag{0.4}$$

for some closed rectangle R. (Here "Int" stands for "interior of".)

3. The <u>volume</u>[2] of a closed rectangle R given by (0.3) is the number $|R|$ defined by
$$|R| = \prod_{j=1}^{d}(b_j - a_j)\,. \tag{0.5}$$

4. The <u>volume</u> of a rectangle S is the volume $|R|$ of any closed rectangle R such that (0.4) holds. (In particular, the volume of the empty rectangle is 0.)

NOTES: The result of this problem follows trivially from the subadditivity of Lebesgue measure, *provided one knows that the volume $|R|$ of a rectangle is equal to its Lebesgue measure $\mu_d(R)$*. But the proof of the fact that $\mu_d(R) = |R|$ depends on inequality (0.2). So ***you are not allowed to justify inequality (0.2) by invoking the subadditivity of Lebesgue measure***, because you don't know that $\mu_d(R) = |R|$.

In class, I outlined a particular way to do this problem. You are free you use this approach to do the problem, or to propose a different approach. Anything you do is fine, as long as it rigorous and written clearly and precisely.

If you find the combinatorial issues pertaining to the proof too complicated, I suggest you try first the case when $d = 2$. Once you understand this situation, the general case will be clear, and the only remaining difficulty will be how to choose the appropriate notations in order to say what you want to say in precise mathematical language.

Problem 6. ***Prove*** the following statement: If A, B are sets, A is infinite, and $\mathrm{card}(B) \le \mathrm{card}(A)$, then $\mathrm{card}(A \cup B) = \mathrm{card}(A)$. (HINT: You may find it convenient to use the result of the last problem of Homework 1.)

[2]When $d = 1$ we use the word "length" rather than "volume", and when $d = 2$ we use the word "area".