MATHEMATICS 501 — FALL 2016

Theory of functions of a real variable I

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HOMEWORK ASSIGNMENTS NO. 5 AND 6, DUE ON FRIDAY, NOVEMBER 4 AND FRIDAY, NOVEMBER 11, RESPECTIVELY

Homework No. 5 consists of Problems 1,2,3 and 4. Homework No. 6 consists of Problems 5, 6, 7 and 8.

Problem 1. In this problem

a. d is a natural number.

b. X is $\mathbb{R}^d$.

c. If $\|x\| = (x_1, \ldots, x_d) \in X$, then the length of x is the number $\|x\|$ given by

$$\|x\| = \sqrt{\sum_{k=1}^{d} x_k^2}\,.$$

d. m (or m_d) is Lebesgue measure on $\mathbb{R}^d$. (So m is defined on $\mathcal{L}_d$, where $\mathcal{L}_d$ is the σ-algebra of all Lebesgue-measurable subsets of X.

e. If $S \subset X$, $\alpha \in \mathbb{R}$, and $\alpha \neq 0$, then the set αS is defined by

$$\alpha S = \{x \in X : (\exists y)(y \in X \wedge x = \alpha y\}\,.$$

f. If $x \in X$ and $R \geq 0$, then $B_d(x, R)$ is the set $\{y \in X : \|y - x\| < R\}$ (that is, $B_d(x, R)$ is the d-dimensional open ball centered at x and having radius R).

1. ***Prove***, using the construction of d-dimensional measure given in class, that if S is any subset of X, $\alpha \in \mathbb{R}$, and $\alpha \neq 0$, then

   i. αS is Lebesgue measurable if and only if S is, and in that case

$$m(\alpha S) = |\alpha|^d m(S)\,.$$

ii. There exists a constant[1] C_d such that, if $x \in X$ and $R \geq 0$, then

$$m\Big(B_d(x,R)\Big) = C_d R^d\,.$$

2. For real (possibly negative) numbers a, b, define a function $f_{a,b} : X \mapsto \mathbb{R}$ by letting

$$f_{a,b}(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \|x\|^a & \text{if} \quad 0 < \|x\| < 1 \\ \|x\|^b & \text{if} \quad 1 \leq \|x\| \end{cases}$$

Determine exactly the set of values of a, b for which $f_{a,b}$ is integrable. (Your answer should have the form "if a is such that ... and b is such that ... then $f_{a,b}$ is integrable, and otherwise it is not". The precise conditions will depend on d.) HINT: First determine the measures of sets of the form $B(0,R) - B(0,r)$ for $0 < r < R$. Also, you will find it useful to remember for which values of σ the series $\sum_{n=1}^{\infty} n^\sigma$ converges.

Problem 2. This problem deals with a proof that appeared in the exam. An important point about this proof is that it requires thr Axiom of choice, but almost nobody seems to have noticed that in the exam.

What I am going to do is this: first, I will give you a correct (and, I hope, well written) proof of a fact that's very similar to (but not exactly the same as) the one that appeared in the exam problem, and then I will ask you to write a detailed proof of the fact that appeared in the problem.

So first I am going to prove: *The union of a countably infinite set of countably infinite sets is a countably infinite set.*

Proof: Let A be a countably infinite set such that every member of A is a countably infinite set. Since A is countably infinite, there exists a bijection from $\mathbb{N}$ onto A. Pick one such bijection and call it F.

For every $n \in \mathbb{N}$, $F(n)$ is a member of A, so $F(n)$ is a countably infinite set. So, if we define a set B_n, for each $n \in \mathbb{N}$, by letting B_n be the set of all bijections from $\mathbb{N}$ to $F(n)$, it follows that the set B_n, for each n, is nonempty (because $F(n)$ is countably infinite, so a bijection from $\mathbb{N}$ to $F(n)$ exists). Hence $(B_n)_{n\in\mathbb{N}}$ is an indexed family of nonempty sets. The Axiom of Choice implies that there exists an indexed family $(b_n)_{n\in\mathbb{N}}$ such that $b_n \in B_n$ for each $n \in \mathbb{N}$. (In other words, there exists a "choice function" that picks a

[1]This constant is, of course, the volume of the d-dimensional unit ball. We will see how to compute C_d later, and get a nice formula involving the Gamma function.

member[2] b_n of B_n for each n.) Since such an indexed family exists, pick one and call it $\mathbf{b}$, and write $\mathbf{b} = (b_n)_{n\in\mathbb{N}}$. Now define a function $G : \mathbb{N}\times\mathbb{N} \mapsto \bigcup A$ by letting

$$G(n,m) = b_n(m) \quad \text{for} \quad (n,n) \in \mathbb{N}\times\mathbb{N}\,.$$

Then G is a map from $\mathbb{N}\times\mathbb{N}$ to $\bigcup A$, because for each n, m $b_n(m)$ belongs to $F(n)$ (since b_n is is a bijection from $\mathbb{N}$ to $F(n)$) and then $b_n(m)$ belongs to $\bigcup A$ (because $F(n) \in A$, so $F(n) \subset \bigcup A$). Furthermore. G maps $\mathbb{N}\times\mathbb{N}$ onto $\bigcup A$. (Reason: if $x \in \bigcup A$ we may pick $X \in A$ such that $x \in X$, and then we may pick $n \in \mathbb{N}$ such that $X = F(n)$, because F maps $\mathbb{N}$ onto A; hence $x \in F(n)$ and, since b_n maps $\mathbb{N}$ onto $F(n)$, it follows that we may pick $m \in \mathbb{N}$ such that $x = b_n(m)$, so $x = G(n,m)$.)

Unfortunately, the map from $\mathbb{N}\times\mathbb{N}$ that we have constructed need not be one-to-one (because the members of A need not be pairwise disjoint). It would not be too hard to use G to construct a true bijection from $\mathbb{N}\times\mathbb{N}$ onto $\bigcup A$. However, it is easier to do something else.

First, pick a bijection H from $\mathbb{N}$ onto $\mathbb{N}\times\mathbb{N}$. (For example, let $K(n,m) = 2^{n-1}(2m-1)$. Then K is one-to-one and onto $\mathbb{N}$, because every natural number can be written uniquely as $2^{n-1}(2m-1)$. So K is a bijection and we may take H to be K^{-1}.) Then the composite map $G \circ H$ is a map from $\mathbb{N}$ onto $\bigcup A$. So for each $x \in \bigcup A$ the set $H^{-1}(x) = \{n \in \mathbb{N} : H(n) = x\}$ is nonempty. Hence the family $(H^{-1}(x))_{x\in\bigcup A}$ is an indexed family of nonempty sets. So by the Axiom of Choice[3] we may pick a function L with domain $\bigcup A$ such that $L(x) \in H^{-1}(x)$ for each $x \in \bigcup A$. Then L satisfies $H(L(x)) = x$ for

[2] ***This is why we need the Axiom of Choice for this proof!*** Each set $F(n)$ is countably infinite, so for each n there exists a bijection from $F(n)$ onto $\mathbb{N}$. Also, there is a logical rule that says: "every time there exists an x such that something about x is true, you are allowed to pick one and give it a name". So, for example, since there exists a bijection from $F(1)$ to $\mathbb{N}$, you are allowed to pick one and call it b_1. You can also pick a bijection from $F(2)$ to $\mathbb{N}$ and call it b_2. And you can pick a bijection from $F(3)$ to $\mathbb{N}$ and call it b_3. But you cannot go on like this and use the words "pick b_n for every n", because to do that you would have to write infinitely many statements, one for each n. And you cannot do that because: (a) a proof is, among other things, a finite sequence of statements. (b) a statement is, among other things, a finite sequence of symbols. "Picking b_n for every n" would require writing an infinite conjunction of statements, and that is not allowed in a proof. So, if you want to use the logical rule that allows you to pick ***one thing***, you need to know that a thing of the kind you want exists. Precisely, you need to know that "there exists a sequence $\mathbf{b} = (b_n)_{n=1}^{\infty}$ such that $b_n \in B_n$ for each $n \in \mathbb{N}$." Once you know this, you are allowed to pick one. But, in order to know this, you need the Axiom of Choice.

[3] This use of the Axiom of Choice could easily be avoided. I let you figure out how.

each $x \in \bigcup A$, and this implies that L is one-to-one (because if $L(x) = L(x')$ then $x' = H(L(x')) = H(L(x)) = x$).

So we have constructed a one-to-one map from $\bigcup A$ to $\mathbb{N}$. It is easy to construct a one-to-one map from $\mathbb{N}$ to $\bigcup A$. (Let $M(m) = b_1(m)$ for $m \in \mathbb{N}$. Then M maps $\mathbb{N}$ into $F(1)$, which is a subset of $\bigcup A$, so $M : \mathbb{N} \mapsto \bigcup A$. And M is one-to-one because b_1 is.)

Since there exist one-to-one maps from $\bigcup A$ to $\mathbb{N}$ and from $\mathbb{N}$ to $\bigcup A$, the Cantor-Schroeder-Bernstein theorem implies $\text{card}(\mathbb{N}) = \text{card}(\bigcup A)$. So $\bigcup A$ is countably infinite. **Q.E.D.**

Using exactly the same pattern as in the above proof, ***prove*** *the following two theorems:*

Theorem 1. If A is a countable set of countable sets, then the union $\bigcup A$ is countable[4].

Theorem 2. If A is a set such that $\text{card}(A) = \text{card}(\mathbb{R})$ and every member X of A satisfies $\text{card}(X) = \text{card}(\mathbb{R})$, then $\text{card}(\bigcup A) = \text{card}(\mathbb{R})$.

Problem 3. Book, problems 19, 20 and 21 on page 59.

Problem 4. ***Compute***, for each real number α, the limit

$$\lim_{n\to\infty} \int_0^\infty \Big(1 + \frac{x}{n}\Big)^n e^{\alpha x} dx\,,$$

and justify your calculation. (The limit will of course depend on α, and could be infinite, or, who knows, might not exist, for some values of α.)

Problem 5. (This is a stronger version of Problem 22 on page 59 of the book.) Let A be an infinite set. Let μ be counting measure on A. (This means, precisely, that the σ-algebra of measurable subsets of A is the set of

[4]This question appeared in the exam, and everybody got it at least partly wrong, for two reasons. First, they failed to point out the use of the Axiom of Choice, which in this course is an important omission. Second, almost everybody got the definition of "countable" wrong. Recall that the official definition of "countable" in this course, following the book (see page 7), and as explained in detail in class, is that "countable" means "finite or countably infinite". (So a countably infinite set is a set S such that $\text{card}(S) = \text{card}(\mathbb{N})$, whereas a countable set is a set S such that $\text{card}(S) \leq \text{card}(\mathbb{N})$.) ***In the exam, most people got this wrong, and said that "countable" means the same as "countably infinite"!*** If you are one of those who got this wrong, you should repeat at home 50 times the mantra: ***in this course, "countable" means "finite or countably infinite"***.

all subsets of A, and the measure $\mu(S)$ of a subset S is n if S is a finite set of cardinality n, and is $+\infty$ if the set S is infinite.)

If $\mathbf{x} = (x_a)_{a\in A}$ is a family of real numbers, or of complex numbers, indexed by A, we would like to define the "sum" $\sum \mathbf{x}$, or $\sum_{a\in A} x_a$. We actually do this as follows: first, we define $\sum_{a\in A} x_a$, if all the x_a are nonnegative real numbers, to be the supremum of the sums $S_F(\mathbf{x}) = \sum_{a\in F} x_a$, ranging over the set of all finite subsets F of A. That is[5],

$$\sum_{a\in A} x_a \stackrel{\mathrm{def}}{=} \sup\Big\{\sum_{a\in F} x_a : F\in\mathcal{P}_{fin}(A)\Big\}\,, \tag{0.1}$$

where $\mathcal{P}_{fin}(A)$ is the set of all finite subsets of A. We then observe that the sum $\sum_{a\in A} x_a$ is always defined, as long as all the x_a are nonnegative reals, and is a nonnegative extended real number, which can be $+\infty$. Next, we define $\sum_{a\in A} x_a$ if all the x_a are real numbers, by writing $x_a = x_{a,+} - x_{a,-}$, where $x_{a,+} = \max(x_a, 0)$ and $x_{a,-} = -\min(x_a, 0)$, so that $x_a = x_{a,+} - x_{a,-}$, and $|x_a| = x_{a,+} + x_{a,-}$. (Notice that an alternative way to define $x_{a,+}$ and $x_{a,-}$ is by letting $x_{a,+} = \frac{|x_a|+x_a}{2}$, $x_{a,-} = \frac{|x_a|-x_a}{2}$.) We then define the sum $\sum_{a\in A} x_a$ to be the difference $\sum_{a\in A} x_{a,+} - \sum_{a\in A} x_{a,-}$, with the proviso that $\sum_{a\in A} x_a$ is ***not*** defined when $\sum_{a\in A} x_{a,+} = +\infty$ and $\sum_{a\in A} x_{a,-} = +\infty$. Finally, we define $\sum_{a\in A} x_a$ if all the x_a are complex numbers, by writing $x_a = \xi_a + i\eta_a$, where the ξ_a and η_a are real numbers, and defining $\sum_{a\in A} x_a$ to be $\sum_{a\in A} \xi_a + i\sum_{a\in A} \eta_a$, with the proviso that $\sum_{a\in A} x_a$ is ***not*** defined when one of the sums $\sum_{a\in A} \xi_a$, $\sum_{a\in A} \eta_a$ is not defined. Finally, we define the family $\mathbf{x}$ to be summable if $\sum_{a\in A} x_a$ is defined and finite. (Equivalently, $\mathbf{x}$ is summable iff the four sums $\sum_{a\in A} \xi_{a.+}$, $\sum_{a\in A} \xi_{a.-}$, $\sum_{a\in A} \eta_{a.+}$, $\sum_{a\in A} \eta_{a.-}$ are finite.)

Now observe that an indexed family $\mathbf{x} = (x_a)_{a\in A}$ is exactly the same as a function with domain A. (The only difference is that when we think of $\mathbf{x}$ as a function we write $\mathbf{x}(a)$ for the value of $\mathbf{x}$ at a, and when we think of $\mathbf{x}$ as an indexed family we write x_a instead.)

1. ***Prove*** that the sum $\sum \mathbf{x}$, as defined above, is exactly the same as the integral $\int \mathbf{x}\, d\mu$.

2. ***Deduce*** from the result of Part 1 the following properties of infinite sums:

[5] "$\stackrel{\mathrm{def}}{=}$" means "equals by definitions".

a. A necessary condition for a family $\mathbf{x} = (x_a)_{a\in A}$ of real or complex numbers to be summable is that the set $\{a \in A : x_a \neq 0\}$ be countable.

b. If $\mathbf{x}$ is a family $(x_a)_{a\in A}$ of real or complex numbers, then $\mathbf{x}$ is summable if and only if $\sum_{a\in A} |x_a| < \infty$.

b. If $(\mathbf{x}_n)_{n=1}^{\infty}$ is a sequence of families $\mathbf{x}_n = (x_{a,n})_{a\in A}$ of real or complex numbers, and $\mathbf{x}_\infty = (x_{a,\infty})_{a\in A}$ is a family of real or complex numbers, such that

$$\lim_{n\to\infty} x_{a,n} = x_{a,\infty} \quad \text{for every} \quad a \in A\,, \tag{0.2}$$

then

$$\lim_{n\to\infty} \sum_{a\in A} x_{a,n} = \sum_{a\in A} x_{a,\infty} \tag{0.3}$$

if one of the following conditions holds:

I. the $x_{a,n}$ are real numbers, $x_{a,n} \leq x_{a,n+1}$ for every $n \in \mathbb{N}$ and every $a \in A$, and there exists a summable family $(s_a)_{a\in A}$ of reals such that

$$x_{a,n} \geq s_a \quad \text{for every} \quad n \in \mathbb{N}\,,\ a \in A\,.$$

II. there exists a summable family $(s_a)_{a\in A}$ of reals such that

$$|x_{a,n}| \leq s_a \quad \text{for every} \quad n \in \mathbb{N}\,,\ a \in A\,.$$

d. The conclusion (0.3) can fail even when (0.2) holds, if both conditions I and II above are violated.

3. ***Deduce*** from the results of Parts 1 and 2 the following properties of infinite sums over a countably infinite[6] set:

a. If A is countably infinite, then a family $\mathbf{x} = (x_a)_{a\in A}$ of real or complex is summable if and only if the limit

$$\lim_{n\to\infty} \sum_{a\in F_n} x_a \tag{0.4}$$

[6] I put the condition that A be countably infinite because if A is uncountable then there do not exist any exhausting sequences of finite subsets of A.

exists and is finite[7] for every exhausting sequence $\mathbf{F} = (F_n)_{n=1}^{\infty}$ of finite subsets of A, and the limit is the same for all exhausting sequences. (An exhausting sequence of subsets of A is a sequence $\mathbf{F} = (F_n)_{n=1}^{\infty}$ of subsets of A which is monotonically increasing, in the sense that $F_n \subset F_{n+1}$ for all n, and in addition satisfies $\bigcup_{n=1}^{\infty} F_n = A$.)

b. Furthermore, if the condition for summability of Part a holds, then the limit (0.4) is the sum $\sum \mathbf{x}$. (HINT FOR PARTS a AND b: First prove that if the x_a are all nonnegative reals then the supremum of (0.1) is equal to the limit of (0.4) for every exhausting sequence $\mathbf{F}$ of finite subsets of A[8]. Using this, the rest should be easy, except for one point that requires some work.)

4. ***Deduce*** from the results of Parts 1, 2 and 3, the following properties of infinite sums over a $\mathbb{N}$. (So in this part A is $\mathbb{N}$, and a family $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is just a sequence $\mathbf{x} = (x_n)_{n=1}^{\infty}$.)

   a. For a sequence $(x_n)_{n\in\mathbb{N}}$ of real or complex numbers, the following conditions are equivalent:

      i. the sequence $(x_n)_{n\in\mathbb{N}}$ is summable,
      ii. $\sum_{n\in\mathbb{N}} |x_n| < \infty$,
      iii. $\sum_{n=1}^{\infty} |x_n| < \infty$ (meaning "$\lim_{N\to\infty} \sum_{n=1}^{N} |x_n| < \infty$"),
      iv. $\sum_{n=1}^{\infty} x_{\pi(n)}$ exists for every permutation[9] $\pi : \mathbb{N} \mapsto \mathbb{N}$, and the sum is independent of π. (HINT: To every permutation π you can assign an exhausting sequence $(F_n)_{n=1}^{\infty}$ by letting $F_n = \{\pi(1), \pi(2), \ldots, \pi(n)\}$. Conversely, even though not every exhausting sequence arises in this way from a permutation, given an exhausting sequence $\mathbf{F} = (F_n)_{n=1}^{\infty}$ you can make a few simple changes to $\mathbf{F}$ (such as eliminating repeated entries, and adding a few extra sets between F_n and F_{n+1}) so as to

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[7]For complex numbers, this means that the limits of the real parts and the imaginary parts are finite.

[8]You can do this with a direct argument or by using the Monotone Convergence Theorem. Naturally, for this course I prefer the latter, because this is a course on integration theory.

[9]A permutation of a set S is a map $f : S \mapsto S$ which is bijective, i.e., one-to-one and onto S.

construct an exhausting sequence that does arise from a permutation.) NOTE: *This result is something you must have seen in one of your Calculus courses: A series* $\sum_{n=1}^{\infty} x_n$ *has the property that no matter how you rearrange the terms of the series the resulting series is convergent, and the sum is the same for all rearrangements, if and only if the series is absolutely convergent.*

5. ***Deduce*** from the results of Parts 1, 2, 3 and 4, the following properties of "double series", i.e., sums over $\mathbb{N} \times \mathbb{N}$:

   a. If a double sequence $(x_{m,n})_{(m,n)\in\mathbf{N}\in\mathbf{N}}$ of real or complex numbers satisfies $\sum_{(m,n)\in\mathbf{N}\times\mathbf{N}} |x_{m,n}| < \infty$, then the limits

   $$\lim_{N\to\infty} \sum_{m=1}^{N} \sum_{n=1}^{N} x_{mn}$$

   and

   $$\lim_{N\to\infty} \sum_{m=1}^{N-1} \Big(\sum_{n=1}^{N-m-1} x_{mn} \Big)$$

   exist and are equal. (HINT: Associate those two limits with two different exhausting sequences.)

   b. If $(x_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}}$ are two summable sequences of real or complex numbers, then the double sequence $(z_{m,n})_{(m,n)\in\mathbf{N}\in\mathbf{N}}$ defined by letting $z_{m,n} = x_m y_n$ is summable, and

   $$\sum_{(m,n)\in\mathbf{N}\times\mathbf{N}} z_{m.n} = \Big(\sum_{n\in\mathbf{N}} x_n \Big) \Big(\sum_{n\in\mathbf{N}} y_n \Big) .$$

   Furthermore, if we define the "product series" by letting

   $$p_n = \sum_{m=1}^{n} x_m y_{n+1-m}$$

   (so, for example, $p_4 = x_1y_4 + x_2y_3 + x_3y_2 + x_4y_1$) then the sequence $(p_n)_{n=1}^{\infty}$ is summable and

   $$\sum_{n\in\mathbf{N}} p_n = \Big(\sum_{n\in\mathbf{N}} x_n \Big) \cdot \Big(\sum_{n\in\mathbf{N}} y_n \Big) .$$

6. Using the definition
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
(that you may rewrite if you wish as $e^z = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!}$) the following are true:

   a. For each $z \in \mathbb{C}$ the series for e^z is summable,
   b. For each $z, w \in \mathbb{C}$ the product series of the e^z series and the e^w series is the series for e^{z+w}.
   c. $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$.

Problem 6. Book, problem 28 on page 60.

Problem 7. Book, problem 29 on page 60. (NOTE: Since we talked in class about the Gamma function, defined by letting
$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \qquad \text{for} \quad z \in \mathbb{C}\,,\ \mathrm{Re}\, z > 0\,,$$
let me point out that the first of the two results of this problem shows that $\Gamma(n+1) = n!$ for every natural number n, so ***the Gamma function is an extension of the function*** $\mathbb{N} \ni n \mapsto (n-1)!$ ***to the complex numbers (or at least to the half-plane*** $H = \{z : \mathrm{Re}\, z > 0\}$***)***. Another way of proving this is by proving first, using integration by parts, the fact that $\Gamma(z+1) = z\Gamma(z)$ whenever $z \in H$, and then, starting with $\Gamma(1) = 1$, proving that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$ by induction.)

Problem 8. Book, problem 30 on page 60. (If you are ambitious, since you know the Gamma function, you may guess that $\lim_{k\to\infty} \int_0^k x^a \Big(1 - \frac{x}{k}\Big)^k dx = \Gamma(a+1)$ for every nonnegative real number a, and perhaps even for every complex $a \in H$. Try to prove that.)