MATHEMATICS 501 — FALL 2016

Theory of functions of a real variable I

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HOMEWORK ASSIGNMENT NO. 8, DUE ON TUESDAY, NOVEMBER 29

The six problems in this list are recommended for you to do. The problems you are asked to hand in are Nos. 1, 4, 5, and 6 (minus the optional parts).

Problem 6 looks very long, but this is because I put in many explanations and very detailed hints. I could have written it in Folland style, and then it would have looked very short.

Problem 1. Book, problem 53, page 76. (NOTE: This is a very important result.)

Problem 2. Book, problem 57, page 77.

Problem 3. Book, problem 62, page 80.

The purpose of Lebesgue integration theory is not to prove theorems about Lebesgue integrals. It is to develop a set of tools that can used to prove results in other areas of mathematics. In the following series of three problems we discuss an example, namely, the construction of a solution

$$(0,\infty)\times\mathbb{R}^d \ni (t,x)\mapsto u(t,x)$$

of the heat equation,

$$\frac{\partial u}{\partial t}=\frac{1}{2}\Delta u$$

with initial condition

$$u(0,x)=f(x)\ \text{ for }\ x\in\mathbb{R}^d\,,$$

where f is a given function in $L^1(\mathbb{R}^d)$.

The key results of this series of problems are:

A. *Problem 6, Part 6, which says that the function u^f constructed in Part 1 of the problem is a solution of the heat equation,*

B. *Problem 6, Part 8, which says that the function $u^f(t,x)$ "satisfies the initial condition $u^f(0,x) = f(x)$", in the sense that*

$$\lim_{t\downarrow 0} u^f(t,x) = f(x)\,, \tag{0.1}$$

provided that the limit in (0.1) is interpreted in the L^1 sense, that is,

$$\lim_{t\downarrow 0} \int_{\mathbf{R}^d} |u^f(t,x) - f(x)| dm_d(x) = 0\,.$$

In addition,

C. *Part 8 provides us with an explicit way (called "heat kernel regularization") to "regularize" L^1 functions, i.e., to approximate L^1 functions in L^1 by functions of class[1] C^∞.*

In Problems 4,5 and 6,

1. *d is a fixed natural number,*

2. *$\mathcal{L}_d$ is the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^d$,*

[1] A function of class C^1 on $\mathbb{R}^d$ is a function $v : \mathbb{R}^d \mapsto \mathbb{C}$ such that the partial derivatives $\frac{\partial v}{\partial x_j}$, for $j \in \{1,\ldots,d\}$, exist at every point x of $\mathbb{R}^d$ and are continuous functions of x. A function of class C^2 is a function of class C^1 all whose first-order partial derivatives are of class C^1. And, in general, one defines "function of class C^k" inductively, as follows: a function of class C^k, if k is a natural number, is a function of class C^1 all whose first-order partial derivatives are functions of class C^{k-1}. And, finally, a function of class C^∞ is a function which is of class C^k for every $k \in \mathbb{N}$.

3. *m_d is d-dimensional Lebesgue measure,*

4. *"$L^1(\mathbb{R}^d)$" means "$L^1(\mathbb{R}^d, \mathcal{L}_d, m_d)$". Recall that $L^1(\mathbb{R}^d)$ is a normed space, with norm defined by*

$$\|f\|_{L^1} = \int_{\mathbf{R}^d} |f(x)| dm_d(x) \,,$$

and then $L^1(\mathbb{R}^d)$ is a metric space, with distance function $\rho : L^1(\mathbb{R}^d) \times L^1(\mathbb{R}^d) \mapsto \mathbb{R}$ defined by

$$\rho(f,g) = \|f-g\|_{L^1} = \int_{\mathbf{R}^d} |f(x)-g(x)| dm_d(x) \,.$$

5. *Δ_d is the d-dimensional Laplace operator, given by*

$$\Delta_d = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2} \,.$$

This means that, for a function $v : \mathbb{R}^d \mapsto \mathbb{C}$ of class C^2,

$$\Delta_d v = \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + \cdots + \frac{\partial^2 v}{\partial x_d^2} \,.$$

6. *If U is an open subset of $\mathbb{R} \times \mathbb{R}^d$, a function $u : U \mapsto \mathbb{C}$ is a <u>solution of the heat equation</u> on U if it is of class[2] C^2 on U and satisfies*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta_d u \ \text{ on } \ U \,.$$

[2]It is possible to define a notion of "solution of the heat equation" for functions that in principle are not of class C^2 or even of class C^1 or even continuous. But one can then prove that the heat operator is "hypoelliptic", meaning that every solution of the heat equation in this generalized sense is in fact of class C^∞.

7. *For each $T \in \mathbb{R}$, we use $H_+^d(T)$ and $\bar{H}_+^d(T)$ to denote, respectively, the open and closed half-spaces of $\mathbb{R}^{d+1}$ defined by*

$$\begin{aligned} H_+^d(T) &= \{(t,x) : t > T \text{ and } x \in \mathbb{R}^d\}\,, & (0.2)\\ \bar{H}_+^d(T) &= \{(t,x) : t \geq T \text{ and } x \in \mathbb{R}^d\}\,. & (0.3) \end{aligned}$$

8. *K_d is the d-dimensional heat kernel, that is, the function $K_d : H_d^+(0) \mapsto \mathbb{R}$ defined by*

$$K(t,x) = \frac{1}{2\pi t^{d/2}} e^{-\frac{\|x\|^2}{2t}} \text{ for } x \in \mathbb{R}^d\,,\; t > 0\,.$$

9. *For each multiindex[3] $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^{d+1}$, we write*

$$\partial^\alpha = \frac{\partial^{\alpha_0}}{\partial t^{\alpha_0}} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$$

so ∂^α is a differential operator acting on functions of the $d+1$ variables $t, x_1, \ldots, x_d$.

Problem 4. *(The result of this problem is an important fact in its own right, but here we need it as a lemma that will be used in problem 6.)*

The translations τ_a, for $a \in \mathbb{R}^d$, are defined on Page 71 of the book. ***Prove*** that if $f : \mathbb{R}^d \mapsto \mathbb{C}$ is a Lebesgue integrable function, then the map $\Phi_f : \mathbb{R}^d \mapsto L^1(\mathbb{R}^d)$ defined by

$$\Phi_f(a) = f \circ \tau_a \qquad \text{for} \quad a \in \mathbb{R}^d$$

is continuous, as a map from the metric space $\mathbb{R}^d$ to the metric space $L^1(\mathbb{R}^d)$. (Continuity of the map Φ_f means $L^1\text{-}\lim_{b\to a} \Phi_f(b) = \Phi_f(a)$, that is,

$$\lim_{b\to a} \|f \circ \tau_b - f \circ \tau_a\|_{L^1} = 0$$

[3] $\mathbb{Z}_+$ is the set of all nonnegative integers, so $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Then $\mathbb{Z}_+^{d+1}$ is the set of all $d+1$-tuples of nonnegative integers.

or, equivalently,

$$\lim_{b\to a}\int_{\mathbf{R}^d}|f(x+b)-f(x+a)|dm_d(x)=0\,,$$

for every $a\in\mathbb{R}^d$.) HINT: First show that, thanks to the translation invariance of m_d, it suffices to show that

$$\lim_{a\to 0}\int_{\mathbf{R}^d}|f(x+a)-f(x)|dm_n(x)=0\,.$$

Then prove the result for compactly supported[4] continuous functions. Then use Theorem 2.41, together with the translation invariance of Lebesgue measure (Theorem 2.42) to prove continuity of Φ_f for general $f\in L^1$.

Problem 5. *(The result of this problem is just a lemma needed for Problem 6.)* ***Prove*** that if $\psi:\mathbb{R}^d\mapsto\mathbb{C}$ is a bounded measurable function which is continuous at 0, then

$$\lim_{\sigma\downarrow 0}\int_{\mathbf{R}^d}|\psi(\sigma h)-\psi(0)|e^{-\frac{\|h\|^2}{2}}dm_d(h)=0\,. \tag{0.4}$$

(HINT: Let I_σ be the integral of (0.4). Split I_σ into two parts, $I_{\sigma,1}$ and $I_{\sigma,2}$ as follows: $I_{\sigma,1}$ is the integral over the set $\{x\in\mathbb{R}^n:\|x\|\le\beta\}$, and $I_{\sigma,2}$ is the integral over the set $\{x\in\mathbb{R}^n:\|x\|>\beta\}$. Then, roughly—and it is your job to do this carefully and rigorously—$I_{\sigma,2}$ can be made small by taking β large, and then, for a fixed β, $I_{\sigma,1}$ can be made small for small enough σ, because of the continuity of ψ at 0.)

Problem 6. If $f:\mathbb{R}^d\mapsto\mathbb{C}$ is a Lebesgue integrable function, we define, for $t>0$,

$$f_t(x)=(2\pi t)^{-d/2}\int_{\mathbf{R}^d}f(y)e^{-\frac{\|y-x\|^2}{2t}}dm_d(y),, \tag{0.5}$$

that is,

$$f_t(x)=\int_{\mathbf{R}^d}f(y)K_d(t,y-x)dm_d(y)\,, \tag{0.6}$$

and we also write

$$u^f(t,x)=f_t(x)\,. \tag{0.7}$$

[4] A function $g:\mathbb{R}^d\mapsto\mathbb{C}$ is <u>compactly supported</u> if there exists a compact subset K of $\mathbb{R}^d$ such that $g(x)=0$ for all $x\in\mathbb{R}^d\backslash K$.

(NOTE: We use the notation $f_t(x)$ if we want to regard (0.5) as defining a family $(f_t)_{t\in(0,\infty)}$ of functions on $\mathbb{R}^d$, indexed by t. And we use $u^f(t,x)$ when we want to regard the right-hand side of (0.5) as a function of t and x, defined on $H_d^+(0)$.)

Prove that

1. The function u^f is continuous on $H_+^d(0)$.
2. For each real number T such that $T > 0$,
   (a) the function u^f is bounded on $H_d^+(T)$, and in fact u^f satisfies a bound
   $$|u^f(t,x)| \leq cT^{-d/2}\|f\|_{L^1}\,,$$
   where c is a constant independent of f and T.
   (b) $u^f(t,x)$ goes to zero as $t + \|x\| \to \infty$ in $H_d^+(T)$.

   (NOTE: In particular, the results of (a) and (b) imply that for each positive t the function f_t is bounded and continuous, and vanishes at infinity[5].
3. u^f is a function of class C^∞ on $H_d^+(0)$.

   HINT:

   a. Now that you know how to differentiate under the integral sign, do it repeatedly.

   b. It will be convenient for you to prove, by induction, that every partial derivative of the heat kernel K_d is of the form PK_d, where P is a polynomial in the variables $x_1, \ldots, x_d$ and $\frac{1}{t}$. For example,
   $$\frac{\partial}{\partial x_1}K(t,x) = -\frac{x_1}{t}K(t,x)\,,$$
   and
   $$\frac{\partial}{\partial t}K(t,x) = \Big(\frac{\|x\|^2}{2t^2} - \frac{d}{2t}\Big)K(t,x)\,,$$
   so in both cases you get a polynomial in $x_1, \ldots, x_n, \frac{1}{t}$ times K.

[5]The sentence "g vanishes at infinity" means "$\lim_{\|x\|\to\infty} g(x) = 0$".

c. Conclude from the result of Part b that every partial derivative of K, of any order, is bounded on $\bar{H}^d_+(T)$, for every T such that $T > 0$.

d. Do not forget that exponential decay always beats polynomial growth.

4. (Optional) If you want to go further with the analysis of Part 3, ***prove***, by carefully analyzing the degrees with respect to $\frac{1}{t}$ of the polynomials P that you get in Part 3, that every partial derivative $\partial^\alpha K$, for every $\alpha \in \mathbb{Z}_+^{d+1}$ satisfies a bound

$$|\partial^\alpha K(t,x)| \le c_\alpha t^{-\nu(\alpha)-\frac{d}{2}} \text{ for } t > 0,\, x \in \mathbb{R}^d\,,$$

where the c_α are constants and, if $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d)$, then

$$\nu(\alpha) = 2\alpha_0 + \alpha_1 + \cdots + \alpha_d\,.$$

(Notice that each differentiation with respect to t counts like two differentiations with respect to the x variables, which should not be too surprising, since $\frac{\partial K}{\partial t} = \frac{1}{2}\Delta_d K$.) This result says that K_t is bounded as long as t is bounded away from zero, and K and all its derivatives blow up as $t \downarrow 0$ like powers of $\frac{1}{t}$.

5. (Continuation of 4. Optional) Prove that u^f and its derivatives satisfy bounds

$$|\partial^\alpha u^f(t,x)| \le c_\alpha t^{-\nu(\alpha)-\frac{d}{2}} \|f\|_{L^1}\,,$$

where the c_α are constants.

5. The equality

$$f_t(x) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(x + \sqrt{t}h) e^{-\frac{\|h\|^2}{2}} dm_d(h) \tag{0.8}$$

holds for each positive t and each $x \in \mathbb{R}^d$. (HINT: Make an appropriate change of variables in the integral of (0.5).)

6. The function u^f is a solution of the heat equation on $H_d^+(0)$. (HINT: Using the result of your differentiation under the integral sign, express $\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta_d\right) u^f$ as an integral involving $\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta_d\right) K$, and verify that K is a solution of the heat equation.)

7. f_t is integrable for each positive t, and

$$\|f_t\|_{L^1} \leq \|f\|_{L^1} \,.$$

(HINT: First conclude from the book's Proposition 2.53 that

$$(2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-\frac{\|h\|^2}{2}} dm_n(h) = 1 \,. \tag{0.9}$$

Then use the translation invariance of Lebesgue measure, the Fubini-Tonelli theorem, and Equation (0.9).)

8. f_t converges to f in L^1 as $t \downarrow 0$ (that is, $\lim_{t\downarrow 0} \|f_t - f\|_{L^1} = 0$). (HINT: Using (0.9), together with (0.8), conclude that

$$f_t(x) - f(x) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} \Big(f(x + \sqrt{t}h) - f(x)\Big) e^{-\frac{\|h\|^2}{2}} dm_n(h) \,,$$

and then, using the notations of Problem 4, show that

$$\|f_t - f\|_{L^1} \leq (2\pi)^{-n/2} \int_{\mathbf{R}^n} \|f \circ \tau_{\sqrt{t}h} - f\|_{L^1} e^{-\frac{\|h\|^2}{2}} dm_n(h) \,, \tag{0.10}$$

Finally, show that the right-hand side of (0.10) goes to zero, using the results of Problems 4 and 5.)