MATHEMATICS 502 — SPRING 2020
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FINAL TAKE-HOME EXAM

*This exam consists of **four** problems.*

You should submit your solutions by sending them by e-mail to

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*not later than **Wednesday, May 13, 2020.***

Problem 1. In this problem,

- The <u>Borel σ-algebra</u> of a topological space X is the σ-algebra generated by the open subsets of X.

- We use $\mathcal{B}(X)$ to denote the Borel σ-algebra of X.

- A <u>Borel probability measure</u> on a topological space X is a nonnegative finite measure $\mu : \mathcal{B}(X) \mapsto \{x \in \mathbb{R} : x \geq 0\}$ such that $\mu(X) = 1$.

Prove that if X is a complete separable metric space and μ is a Borel probability measure on X then μ is "almost concentrated on compact sets", in the following precise sense:

$$\sup\{\,\mu(K) : K \subseteq X\,,\ K \text{ compact}\,\} = 1\,.$$

Problem 2. Using a trick similar to that used by Folland in Problem 13 on Page 254, ***prove*** that

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}\,. \tag{1}$$

Problem 3. Folland, problem 14 of section 8.3, on pages 254-5, on Wirtinger's inequality.

Problem 4. Folland, problem 23 of section 8.3, on pages 256-7, on the Hermite functions.

COMMENTS. The Fourier and Fourier inversion formulas[1]

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(\xi)e^{-2\pi i \xi \cdot x} d\xi\,, \tag{2}$$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i \xi \cdot x} d\xi\,, \tag{3}$$

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[1]Formulas (2) and (3) are interpreted in the usual way: they are valid as written for $f \in L^1$ such that $\hat{f} \in L^1$ (in which case the functions f and $\hat{f}$ are continuous, so the evaluation of these functions at one point makes sense, and the integrands are integrable functions), and then they can be extended to $f \in L^2$ (in which case $\hat{f} \in L^2$ as well) by taking limits in L^2 of functions f_n such that f_n and $\hat{f}_n$ belong to L^1.

imply

$$f(-x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{-2\pi i \xi \cdot x}d\xi\,, \tag{4}$$

i.e.,

$$f(-x) = \hat{\hat{f}}(x)e^{-2\pi i \xi \cdot x}d\xi\,. \tag{5}$$

Hence, if we use $\mathcal{F}$ for the Fourier transformation map in $L^2(\mathbb{R})$ (so that $\mathcal{F}f = \hat{f}$), we have

$$f(-x) = \mathcal{F}\mathcal{F}f(x)\,, \tag{6}$$

and this implies that

$$\mathcal{F}\mathcal{F}\mathcal{F}\mathcal{F} = \mathbb{I}\,, \qquad \text{i.e.,} \qquad \mathcal{F}^4 = \mathbb{I}\,, \tag{7}$$

where $\mathbb{I}$ is the identity map of $L^2(\mathbb{R})$. Furthermore, the Plancherel Theorem says that $\mathcal{F}$ is a unitary map. Hence $\mathcal{F}$ is a unitary map that satisfies $\mathcal{F}^4 = 1$. This implies[2] that the eigenvalues of $\mathcal{F}$ must be $1, i, -1$, and $-i$.

The Hermite functions give us, rigorously, an orthonormal basis of $L^2(\mathbb{R})$ consisting of eigenfuntions for $\mathcal{F}$. Precisely, in the problem we construct a sequence $\{\emptyset_k\}_{k=0}^{\infty}$ of functions $\emptyset_k$ belonging to $L^2(\mathbb{R})$ such that the $\emptyset_k$ form an orthonormal basis of $L^2(\mathbb{R})$ and $\mathcal{F}\emptyset_k = (-i)^k\emptyset_k$ so, as you can see, the $\emptyset_k$ are eigenfunctions for $\mathcal{F}$ for the eigenvalues $1, i, -1$ and $-i$.

It would be nice if we could say that *"the $\emptyset_k$ are the famous <u>Hermite functions</u>.* Unfortunately, we cannot say that, exactly. The Hermite functions, denoted by h_k in Folland's book, are closely related, but not exactly the same as, the eigenfunctions $\emptyset_k$. I think it is important that you understand how they are related, and why they are close but not exactly the same.

The Hermite functions are eigenfunctions of the **Hermite differential operator**[3] S, given by

$$S = -\frac{d^2}{dx^2} + x^2\,, \tag{8}$$

that is[4]:

$$Sf(x) = -f''(x) + x^2 f(x)\,. \tag{9}$$

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[2]The implication is completely rigorous using the spectral theorem and the spectral mapping theorem, but for our purpose here this fact does not matter. So maybe I shoudl have written "suggests" rather than "implies".

[3]In the literature, the formula most commonly used for the Hermite differential operator is $\frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right)$. This, of course, does not change the eigenfuctions, but it changes the eigenvalues, so that the statement you will usually encounter is that the eigenvalues of the Hermite operator are $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$.

[4]S is a **partially defined operator** on $L^2(\mathbb{R})$: Sf is not well defined for every function f in L^2; it's only defined for functions f such that the second derivative f'' exists in some appropriate sense and the function $\mathbb{R} \ni x \mapsto -f''(x) + x^2 f(x)$ is in L^2. This can be made completely rigorous but we do not need to do it here.

As you will show in this problem, the Hermite functions h_k, for $k = 0, 1, 2, \ldots$, satisfy $Sh_k = (2k+1)h_k$, so they are indeed eigenfunctions for S, corresponding to the eigenvalues $1, 3, 5, 7, \ldots$.

How is this related to the Fourier transform $\mathcal{F}$? Roughly speaking, **the Hermite operator S commutes with the Fourier transform operator $\mathcal{F}$**. And, as you know from elementary linear algebra, when two operators commute then they have a common set of eigenfunctions or eigenvectors[5].

So, roughly speaking, what we do in this problem is find the eigenfunctions of S in order to find eigenfunctions of $\mathcal{F}$.

There is, however, one complication. The Hermite operator S does not actually commute with $\mathcal{F}$. It commutes with the "Fourier transform map" $\tilde{\mathcal{F}}$ defined by letting

$$\tilde{\mathcal{F}}f = \tilde{f}, \qquad \text{where } \tilde{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\xi \cdot x}dx\,,$$

which is not exactly the same map as $\mathcal{F}$. (This is, in my opinion, how the Fourier transform should have been defined.)

So, if Folland had used $\tilde{\mathcal{F}}$ rather than $\mathcal{F}$, we would have been able to say things much more simply: the Hermite operator S commutes with $\tilde{\mathcal{F}}$, and the eigenfunctions h_k of S are then also the eigenfunctions of $\tilde{\mathcal{F}}$. But, since we are using $\mathcal{F}$, we cannot quite say that. We have to say instead what Folland says: there is a unitary rescaling map A that conjugates $\mathcal{F}$ and $\tilde{\mathcal{F}}$, in the sense that $A^{-1}\mathcal{F}A = \tilde{\mathcal{F}}$. Since the Hermite functions h_k are eigenfunctions of $\tilde{\mathcal{F}}$, their conjugates $\emptyset_k = Ah_k$ are eigenfunctions of $\mathcal{F}$.

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[5]Here is one rigorous formulation: On a finite-dimensional space, if A and B are commuting linear maps, and all the eigenvalues of A are simple, then the eigenfunctions of A are also eigenfunctions of B. *Proof.* Let $Af = \lambda f$, $f \neq 0$. Let $E = \{h : Ah = \lambda h\}$. Then, if $h \in E$, we have $A(Bh) = B(Ah) = B(\lambda h) = \lambda Bh$, so $Bh \in E$. So E is a B-invariant subspace. Since E is one-dimensional, because λ is simple, we conclude that Bf is a scalar multiple of f, so f is an eigenfunction of B.