MATHEMATICS 502 — SPRING 2020

THEORY OF FUNCTIONS OF A REAL VARIABLE II

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HOMEWORK ASSIGNMENT NO. 1, DUE ON THURSDAY, FEBRUARY 6

Problem 1. ***Give an example*** of a function $f : \mathbb{R} \mapsto \mathbb{R}$ such that

1. the derivative $f'(x)$—that is, the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$—exists for every $x \in \mathbb{R}$,

2. it is not true that

   (&) *for every $a, b \in \mathbb{R}$ such that $a < b$,*

$$f(b) - f(a) = \int_a^b f'(x)dx\,, \tag{1}$$

   where the integral in (1) is a Lebesgue integral.

Problem 2. A sequence $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$ of complex numbers is <u>summable</u> if $\sum_{k\in\mathbb{N}} |x_k| < \infty$ (i.e., if the series $\sum_{k\in\mathbb{N}} x_k$ is absolutely convergent).

Let $\mathbf{y} = (y_k)_{k\in\mathbb{N}}$ be a sequence of complex numbers such that

(*) *For every summable sequence $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$ of complex numbers, the sequence $(x_k y_k)_{k\in\mathbb{N}}$ is summable (i.e., $\sum_{k\in\mathbb{N}} |x_k y_k| < \infty$).*

Prove that $\mathbf{y}$ is bounded (that is, there exists $C \in \mathbb{R}$ such that $|y_k| \le C$ for every $k \in \mathbb{N}$). (COMMENT: This obviously has something to do with the duality between L^1 and L^∞. If you are familiar with the Banach-Steinhaus theorem—a.k.a. Uniform Boundedness Principle, UBP—then this problem is almost trivial. Otherwise, you could find out what the UBP says and then usu it, but I strongly recommend that you do the problem directly, and then you will appreciate the value of the UBP.)

Problem 3. For $1 \le p < \infty$, a sequence $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$ of complex numbers is <u>p-summable</u> if the sequence $(|x_k|^p)_{k\in\mathbb{N}}$ is summable, i.e., if the sum $\sum_{k\in\mathbb{N}} |x_k|^p$ is finite. (So "1-summable" means the same as "summable".)

We define ℓ^p to be the set of all p-summable sequences of complex numbers. For $\mathbf{x} \in \ell^p$, if $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$, define $\|\mathbf{x}\|_{\ell^p}$ to be the number $\Big(\sum_{k\in\mathbb{N}} |x_k|^p\Big)^{1/p}$. Also, define ℓ^∞ to be the set of all bounded sequences $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$ of complex numbers, and for $\mathbf{x} \in \ell^\infty$ define $\|\mathbf{x}\|_{\ell^\infty} = \sup\{|x_k| : k \in \mathbb{N}\}$ if $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$. (COMMENT: you should know how to prove that the ℓ^p are Banach spaces, but I am not asking you to do this as homework.)

Let $1 \leq p \leq \infty$, and let $q = \frac{p}{p-1}$, so $\frac{1}{p} + \frac{1}{q} = 1$. (In particular, $q = \infty$ if $p = 1$ and $q = 1$ if $p = \infty$.)

Prove that, if $\mathbf{y} = (y_k)_{k\in\mathbb{N}}$ is a sequence of complex numbers such that

(**) *for every sequence $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$ of complex numbers such that $\mathbf{x} \in \ell^p$, the sequence $(x_k y_k)_{k\in\mathbb{N}}$ is summable (i.e., $\sum_{k\in\mathbb{N}} |x_k y_k| < \infty$),*

then $\mathbf{y} \in \ell^q$. (COMMENTS: Again, this is related to the duality betwween L^p and L^q, and is trivial if you know the UBP. And, again, I recommend that you do this problem directly, and then you will get to appreciate better the value of the UBP.)

Problem 4. Recall that a <u>real interval</u> is a subset I of $\mathbb{R}$ such that

(#) whenever a, b, c are real numbers such that $a < b < c$, $a \in I$ and $c \in I$, it follows that $b \in I$.

Also, an <u>extended real interval</u> is a subset I of the extended real line[1] $\bar{\mathbb{R}}$ such that

(##) whenever a, b, c are extended real numbers such that $a < b < c$, $a \in I$ and $c \in I$, it follows that $b \in I$.

(COMMENT: If you have not done the following before, do it now at least once, to maake sure you know how to do it; but ***I am not asking you to hand it in***, because this is really an undergraduate math problem: *prove that a subset I of $\mathbb{R}$ is an interval if and only if one of the following conditions holds: (i) $I = \emptyset$, (ii) $I = \{x \in \mathbb{R} : a < x < b\}$ for some $a, b \in \mathbb{R}$ such that $a < b$, (iii) $I = \{x \in \mathbb{R} : a \leq x < b\}$ for some $a, b \in \mathbb{R}$ such that $a < b$, (iv) $I = \{x \in \mathbb{R} : a < x \leq b\}$ for some $a, b \in \mathbb{R}$ such that $a < b$, (v) $I = \{x \in \mathbb{R} : a \leq x \leq b\}$ for some $a, b \in \mathbb{R}$ such that $a \leq b$, (vi) $I = \{x \in \mathbb{R} : a < x\}$ for some $a \in \mathbb{R}$, (vii) $I = \{x \in \mathbb{R} : a \leq x\}$ for some $a \in \mathbb{R}$, (viii) $I = \{x \in \mathbb{R} : x < a\}$ for some $a \in \mathbb{R}$, (ix) $I = \{x \in \mathbb{R} : x \leq a\}$ for some $a \in \mathbb{R}$, (x) $I = \mathbb{R}$.* And then you can do the obvious analogue for extended real intervals.)

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[1] The <u>extended real line</u> is the set $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, where $-\infty$ and $+\infty$ are any two objects that are different and do not belong to $\mathbb{R}$.

1. ***Prove*** that if $(X, \mathcal{A}, \mu)$ is a measure space then for every measurable function $f : X \mapsto \mathbb{C}$ the set $P_f = \{p \in \mathbb{R} : p \geq 1 \text{ and } f \in L^p(X, \mathcal{A}, \mu)$ is an interval. (HINT: Hölder's inequality.)

2. ***Give examples***, for $X = \mathbb{R}$, $\mathcal{A}$ the Borel or Lebesgue σ-algebra, and μ =Lebesgue (or Borel) measure, showing that every extended real interval I such that $I \subseteq [1, +\infty]$ can occur as P_f for some f.

3. ***Prove*** that if $(X, \mathcal{A}, \mu)$ is a finite measure space (that is, $\mu(X) < \infty$) then for every f, if the interval P_f is not empty, then $1 \in P_f$. (That is, if $p \in [1, \infty]$ and $f \in L^p(X\mathcal{A}, \mu)$ then $f \in L^1(X\mathcal{A}, \mu)$.)

4. ***Give examples***, for $X = [0, 1]$, $\mathcal{A}$ the Borel or Lebesgue σ-algebra, and μ =Lebesgue (or Borel) measure, showing that every extended real interval I such that $I \subseteq [1, +\infty]$ and either $I = \emptyset$ or $1 \in I$ can occur as P_f for some f.

5. ***Prove*** that if $(X, \mathcal{A}, \mu)$ is the space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \kappa)$, where $\mathcal{P}(\mathbb{N})$ is the σ-algebra of all subsets of $\mathbb{N}$, and κ is counting measure—defined by letting $\kappa(S) = \sum_{s \in S} 1$—then for every f, if the interval P_f is not empty, then $+\infty \in P_f$. (That is[2], if $\mathbf{x} \in \ell^p$ for some $p \in [1, \infty]$, then $\mathbf{x} \in L^1(X\mathcal{A}, \mu)$.)

6. ***Give examples***, for $(X, \mathcal{A}, \mu) =$ the space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \kappa)$ ccnsidered above, showing that every extended real interval I such that $I \subseteq [1, +\infty]$ and either $I = \emptyset$ or $+\infty \in I$ can occur as P_f for some f.

NOTE: *One of the following two problems is very, very easy. So don't worry if when yo do it it looks "too easy" to you.*

Problem 5. ***Characterize***[3] the measure spaces $(X, \mathcal{A}, \mu)$ for which the property of part 3 of Problem 4 holds (that is, for which $L^p \subseteq L^1$ for every $p \in [1, \infty]$).

Problem 6. ***Characterize*** the measure spaces $(X, \mathcal{A}, \mu)$ for which the property of part 5 of Problem 4 holds (that is, for which $L^p \subseteq L^\infty$ for every $p \in [1, \infty]$).

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[2]Naturally, I am using the fact that $L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \kappa) = \ell^p$.

[3]Yes, I know. If I ask you to characterize the African animals that have very long legs, a very long neck and tiny little horn-lile protuberances on the head, and eat leaves from the top of very tall plants, you can answer "they are exactly the African animals that have very long legs, a very long neck and tiny little horn-lile protuberances on the head, and eat leaves from the top of very tall plants", and this is technically a correct answer. But in this course it gets a zero. I want a simple characterization, such as "they are the giraffes".