MATHEMATICS 502 — SPRING 2020

THEORY OF FUNCTIONS OF A REAL VARIABLE II

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HOMEWORK ASSIGNMENT NO. 2, DUE ON TUESDAY, FEBRUARY 18

Problem 1. *In this problem, "countable" means "finite or countably infinite". That is, a set S is* ***countable*** *if and only if there exists a one-to-one map from S to $\mathbb{N}$.*

Let $X = \mathbb{R}$. Let $\mathcal{A}$ be the set of all subsets S of X such that either S is countable or $X - S$ is countable. Let $\mu : \mathcal{A} \mapsto \mathbb{R} \cup \{+\infty\}$ be the function such that

- $\mu(S)$ is the number of members of S, if S is a finite subset of X,
- $\mu(S) = +\infty$ if $S \in \mathcal{A}$ and S is infinite.

1. ***Prove*** that $\mathcal{A}$ is a σ-algebra, μ is a measure on the measurable space $(X, \mathcal{A})$, and μ is not σ-finite.
2. ***Construct*** a bounded linear functional λ on $L^1(X, \mathcal{A}, \mu)$ such that there exists no function $g \in L^\infty(X, \mathcal{A}, \mu)$ for which $\lambda(f) = \int_X gf \, d\mu$ for every $f \in L^1(X, \mathcal{A}, \mu)$.

Problem 2. If $(X, \mathcal{A}, \mu)$ is a measure space, an <u>atom</u> of $(X, \mathcal{A}, \mu)$ (or of μ) is a set $A \in \mathcal{A}$ such that $\mu(A) > 0$ and there exists no subset B of A such that $B \in \mathcal{A}$ and $0 < \mu(B) < \mu(A)$. An <u>infinite atom</u> is an atom A such that $\mu(A) = +\infty$.

Prove that the canonical map

$$L^\infty(X, \mathcal{A}, \mu) \ni g \mapsto \lambda_g \in L^1(X, \mathcal{A}, \mu)^*$$

given by

$$\lambda_g(f) = \int_X fg \, d\mu \ \text{ for } \ f \in L^1(X, \mathcal{A}, \mu)$$

is one-to-one if and only if $(X, \mathcal{A}, \mu)$ has no infinite atoms.

Problem 3. In this problem, you are asked to analyze in detail a Calculus of Variations minimization problem, and in particular prove a theorem on existence of a solution.

The problem we consider is that of minimizing a "cost functional"

$$J(f) = \int_a^b \Phi(t, f(t))dt + \int_a^b |f'(t)|^p \, dt \tag{1}$$

subject to a constraint $f(a) = \alpha$, $f(b) = \beta$. The goal is to "find" (or at least prove the existence of) a function f_* that minimizes the functional J, in the sense that $J(f_*) \leq J(f)$ for "all functions" f.

Naturally, we cannot truly allow "all functions" because, for example, for a completely arbitrary function the integrals of (1) need not exist. (For example, f might be nowhere differentiable, in which case $f'(t)$ does not exist for any t, or the function $[a, b] \ni t \mapsto \Phi(t, f(t))$ might not be integrable, or even measurable.) So we need to be more precise, make appropriate technical assumptions about Φ, and select an appropriate function space for the functions f.

We will assume:

A1. a, b, α, β, p are real numbers such that $a < b$,

A2. $\Phi : [a, b] \times \mathbb{R} \mapsto \mathbb{R}$ is a function such that

1. the function $[a, b] \ni t \mapsto \Phi(t, x)$ is measurable for every fixed $x \in \mathbb{R}$,
2. the function $\mathbb{R} \ni X \mapsto \Phi(t, x)$ is continuous for every fixed $t \in [a, b]$,
3. $\Phi(t, x) \geq 0$ for every $(t, x) \in [a, b] \times \mathbb{R}$,
4. there exists a nonnegative, integrable function $C : [a, b] \mapsto \mathbb{R}$ such that

$$\Phi(t, x) \leq C(t) \text{ for every } (t, x) \in [a, b] \times \mathbb{R} \,. \tag{2}$$

A3. $p > 1$.

We then let $\mathcal{F}$ (the "feasible set") be the set of all absolutely continuous functions $f : [a, b] \mapsto \mathbb{R}$ such that $f(a) = \alpha$ and $f(b) = \beta$.

1. ***Prove*** that for every function $f \in \mathcal{F}$ the cost $J(f)$ is well defined. Precisely, you should

   i. ***prove*** that if $f : [a, b] \mapsto \mathbb{R}$ is an arbitrary continuous function then the function $[a, b] \ni t \mapsto \Phi(t, f(t))$ is meaasurable[1],

[1] This is not immediately obvious given our technical hypotheses! It requires proof.

ii. ***conclude*** that if $f : [a, b] \mapsto \mathbb{R}$ is an arbitrary continuous function then the function $[a, b] \ni t \mapsto \Phi(t, f(t))$ is integrable[2],

iii. ***conclude*** that if $f \in \mathcal{F}$ then $J(f)$ is well defined[3].

2. Let
$$\theta = \inf \Big\{ J(f) : f \in \mathcal{F} \Big\}. \tag{3}$$
Prove[4] that $0 \leq \theta < +\infty$.

3. ***Prove*** that our problem has a solution. That is, prove that there exists $f \in \mathcal{F}$ such that $J(f) = \theta$.

HINT: You may want to use the Banach-Alaouglu theorem[5], and the Ascoli-Arzelà theorem.

NOTE: The hint does *not* say that you *have* to use Banach-Alaoglu and/or Ascoli-Arzelà. It may very well be possible to prove the result in a different way, not using those theorems. All I know is that ***the proof I am thinking of uses those theorems***, and on that basis I recommend that you use them too, but if you find a different proof that's O.K. too[6].

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[2] This one is completely trivial, but I still want to see the proof.

[3] But $J(f)$ could be $+\infty$, because our definition of $\mathcal{F}$ only guarantees that $f' \in L^1$, and this does not imply that $f \in L^p$.

[4] This one is also completely trivial, but I still want to see the proof.

[5] If X is a normed space then if a sequence $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ in the dual X^* satisfies a bound $(\forall n \in \mathbb{N}) \|x_n\| \leq C$ for some constant C, then $\mathbf{x}$ has a subsequence that converges weak* to a member of X^* of norm $\leq C$. This is a special case of the more general ***Bourbaki-Alaoglu theorem***, valid for general locally convex topological vector spaces and involving nets, rather than sequences.

[6] Or even better!