# MATHEMATICS 502 - SPRING 2020 <br> THEORY OF FUNCTIONS OF <br> A REAL VARIABLE II 

H. J. SUSSMANN

## HOMEWORK ASSIGNMENT NO. 3, DUE ON THURSDAY, FEBRUARY 27

This assignment consists of three problems.

## Problem 1.

1. Prove that if $(S, \mathcal{A}, \mu)$ is a measure space, $1<p<\infty$, and $X=$ $L^{p}(S, \mathcal{A}, \mu)$, then every nonempty, closed, convex subset of $X$ has a member of minimum norm.
2. Prove that it is not true in general that every nonempty, closed, comvex, bounded subset of $X$ has a member of maximum norm.
3. Prove that the result of Part 1 is not true in general for $p=1$.

Problem 2. Let $a, b, \alpha, \beta$ be real numbers, such that $a<b$.
We consider the problem of minimizing the cost functional $f \mapsto J(f)$, given by

$$
J(f)=\int_{a}^{b} L\left(f(t), f^{\prime}(t)\right) d t
$$

in the set $\mathcal{F}_{a, b, \alpha, \beta}$, where $\mathcal{F}_{a, b, \alpha, \beta}$ is the set of all functions $f:[a, b] \mapsto \mathbb{R}$ such that

1. $f$ is absolutely continuous,
2. $\left|f^{\prime}(t)\right| \leq 1$ for almost all $t \in[a, b]$,
3. $f(a)=\alpha$ and $f(b)=\beta$.

The function $\mathbb{R} \times[-1,1] \ni(x, u) \mapsto L(x, u)$ is assumed to satisfy the following continuity and convexity assumptions:
(A1) $L$ is continuous,
(A2) for each fixed $x \in \mathbb{R}$, the function $[-1,1] \ni u \mapsto L(x, u)$ is convex ${ }^{1}$.
Prove that the problem has a solution. That is, prove that thetre exists $f_{*} \in \mathcal{F}_{a, b, \alpha, \beta}$ such that $J\left(f_{*}\right) \leq J(f)$ for every $f \in \mathcal{F}_{a, b, \alpha, \beta}$.

HINTS: You may want to use weak and weak* convergence, AscoliArzelà, and the theorem that in a normed space $X$ a closed convex set is weakly closed. Make an intelligent choice of function spaces to work on. (For example: sometimes it may appear that some space, such as $L^{\infty}$, is the natural choice for a problem, but if you think it over you may discover that it's better to work on some other space, such as $L^{2}$, because it has better properties, such as being reflexive.)

In order to use the convexity assumption, you may find it convenient to make precise and then prove the fact that if $C \in \mathbb{R}$ and $h$ is a fixed function then the set of functions $g$ such that $\int_{a}^{b} L(h(t), g(t) d t \leq C$ is convex.

Problem 3. The purpose of this problem is to show that the convexity assumption of the previous problem is essential.

In this problem we consider exactly the same situation as in Problem 2, but we drop the convexity assumption on $L$. Specifically, we let

$$
L(x, u)=x^{2}+\left(u^{2}-1\right)^{2} .
$$

(It is clear that the functions $u \mapsto L(x, u)$ are not convex.)
Prove that the minimization problem need not have a solution. Specifically, prove that if $\alpha=\beta=0$ then the problem does not have a solution. (HINT: The integral $\int_{a}^{b} f(t)^{2} d t$ wants to be as small possible, and for this purpose it wants to make $f(t)$ as small as possible. The smallest value it can have is 0 , and this value is attained by the functiom $f(t) \equiv 0$. However, the integral $\int_{a}^{b}\left(f^{\prime}(t)^{2}-1\right)^{2} d t$ also wants to be as small as possible, and for this purpose it wants $f^{\prime}(t)^{2}$ to be 1 or near 1 . Obviously, it is not possible to make both integrals equal to zero simultaneously, so there does not exist a function $f \in \mathcal{F}_{a, b, 0,0}$ such that $J(f)=0$. However, one can come arbitarily close to zero by choosing functions $f_{n}$ whose derivative oscillates rapidly between the values 1 and -1 , in such a way that $f_{n}(t)$ remains very small.)

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[^0]:    ${ }^{1}$ This means that, if $u_{1}, u_{2} \in[-1,1]$ and $0 \leq t \leq 1$, then it follows that the inequality $L\left(x, t u_{1}+(1-t) u_{2}\right) \leq t L\left(x, u_{1}\right)+(1-t) L\left(x, u_{2}\right)$ holds.

