# MATHEMATICS 502 - SPRING 2020 <br> THEORY OF FUNCTIONS OF <br> A REAL VARIABLE II <br> H. J. SUSSMANN 

## LIST OF HOMEWORK PROBLEMS FOR THE WEEKS OF MARCH 23-27 AND MARCH 30-APRIL 3

You should submit your solutions to the starred problems by sending them by e-mail to both cr718@math.rutgers.edu and sussmann@math.rutgers.edu. The solutions will not be graded but will be read and you will receive comments.
(I) Folland, pages 177-178, problems *54, 55, *56, 57, *58, *59, 60, 61, *62, *63.
(II) Let $A$ be a set and let $\ell^{2}(A)$ be the $L^{2}$ space of the measure space $\left(A, \mathcal{A}_{A}, c_{A}\right)$ where $\mathcal{A}_{A}$ is the $\sigma$-algebra of all subsets of $A$, and $c_{A}$ is counting measure on $A$. You know from Problem 54 of page 177 that $\ell^{2}(A)$ is a Hilbert space. Let $\varphi: A \mapsto \mathbb{C}$ be a bounded function. Let $M_{\varphi}$ be the operator of multiplication by $\varphi$, that is, the map from $\ell^{2}(A)$ to $\ell^{2}(A)$ given by $M_{\varphi}(f)(a)=\varphi(a) f(a)$ for $a \in A$. Prove that

1. $M_{\varphi}$ is a bounded linear map from $\ell^{2}(A)$ to $\ell^{2}(A)$.
2. $M_{\varphi}$ is a normal operator (that is, $M_{\varphi} M_{\varphi}^{*}=M_{\varphi}^{*} M_{\varphi}$ ), and $M_{\varphi}$ is self-adjoint if and only if $\varphi$ is real-valued (that is, $\varphi(a) \in \mathbb{R}$ for every $a \in A$ ).
*3. $M_{\varphi}$ is a compact linear map if and only if $\varphi$ "vanishes at infinity", in the sense that for every positive $\varepsilon$ there exists a finite subset $F$ of $A$ such that $|\varphi(a)|<\varepsilon$ whenever $a \in A, a \notin F$. (NOTE: if we add to $A$ a "point at infinity" $\infty$, declare a subset $S$ of $A \cup\{\infty\}$ to be open if and only if either (i) $S=\emptyset$ or (ii) $\infty \in S$ and the complement of $S$ is $A$ is finite, then " $\varphi$ vanishes at infinity" is precisely equivalent to " $\lim _{a \rightarrow \infty} \varphi(a)=0$ ".)
HINT: Show that the values of $\varphi$ are the eigenvalues of $M_{\varphi}$, and identify the eigenfunctions (i.e., the eigenvectors). Show that a multiplication operator $M_{\theta}$ on a space $\ell^{2}(B)$ corresponding to a function $\theta$ such that $|\theta|$ is bounded below by some strictly positive $\delta$ is compact if and only if $B$ is finite.)

NOTE: The spectral theorem for compact normal operators says that the converse of the result of (II. $2,{ }^{*} 3$ ) is true: If $H$ is a Hilbert space and $K: H \mapsto H$ is a compact normal operator, then the pair $(H, K)$ is equivalent to the pair $\left(\ell^{2}(A), M_{\varphi}\right)$ for some set $A$ and some bounded function $\varphi$ on $A$ that vanishes at infinity. (Precisely, there exists a Hilbert space isomorphism $\Psi: H \mapsto \ell^{2}(A)$ such that $\Psi^{-1} M_{\varphi} \Psi=K$.)
(III) Folland, page 239, problems 1, 2, 3 and *4.
(IV) Folland, page 248, problems $* 7,8$, and $* 9$.
(V) ${ }^{*}$. Folland, page 254, problem 13.
*2. Use a trick similar to that of the previous problem (with a suitable choice of function $f$ ) to prove that

$$
\sum_{k=1}^{\infty} \frac{1}{k^{4}}=\frac{\pi^{4}}{90}
$$

(*VI) Consider the following general way of defining the "Fourier transform" $\hat{f}$ and the "inverse Fourier transform" $\check{f}$ of a function in $L^{1}(\mathbb{R})$ : we fix two positive real numbers $a, b$, and define

$$
\begin{align*}
& \hat{f}(y)=a \int_{-\infty}^{\infty} f(x) e^{-b x y i} d x  \tag{1}\\
& \check{f}(x)=a \int_{-\infty}^{\infty} f(y) e^{b x y i} d y \tag{2}
\end{align*}
$$

For what values of the parameters $a, b$ does the identity

$$
\begin{equation*}
f(x)=\check{\hat{f}}(-x) \quad \text { for all } \quad x \in \mathbb{R} \tag{3}
\end{equation*}
$$

hold? Prove, in particular, that (3) holds if and only if $b=2 \pi a^{2}$. (Folland chooses $a=1, b=2 \pi$. In my notes I choose $a=\frac{1}{\sqrt{2 \pi}}$ and $b=1$.)

