MATHEMATICS H311 — FALL 2015

Introduction to Mathematical Analysis

H. J. Sussmann

NOTES ON CONNECTED SETS

Contents

1 Connected sets

In this note we show how the general definition of "connected metric space" is related to the definition of "connected set" given in the book, and we discuss several important properties of connected sets.

The book does not talk about general metric spaces, and only works within one metric space, namely, $\mathbb{R}$. We, on the other hand, will discuss connectedness of a general metric space.

Furthermore, a subset of $\mathbb{R}$ is a particular example of a metric space. It follows from this that, once we know what it means for a general metric space to be "connected", this will determine, in particular, what it means for a subset S of $\mathbb{R}$ to be connected.

But, since the book does not talk about general metric spaces, and does not view subsets of $\mathbb{R}$ as metric spaces in their own right, the way the book defines "connected set" is special: instead of looking at a subset S of $\mathbb{R}$ as a metric space, the book uses the fact that S is a subset of $\mathbb{R}$ to give a definition that involves working in $\mathbb{R}$.

We will discuss both definitions of "connected set" and prove that they are equivalent.

1 Connected metric spaces

Roughly, a metric space is said to be "connected" if it cannot be divided into two open parts. Precisely,

Definition 1. Let X be a metric space. We say that X is <u>disconnected</u> if there exist two subsets U, V of X such that

1. U and V are open,
2. U and V are nonempty,
3. $U \cap V = \emptyset$,
4. $U \cup V = X$.

We say that X is <u>connected</u> if it is not disconnected. □

Let us say the same thing in a different way:

Theorem 1. *Let X be a metric space. Then X is disconnected if and only if there exist two subsets A, B of X such that*

(D1) A and B are nonempty,

(D2) $A \cup B = X$,

(D3) $\bar{A} \cup B = \emptyset$ and $A \cup \bar{B} = \emptyset$.

Proof. Suppose X is disconnected. Then we can pick two subsets U, V of X such that U and V are open and nonempty, $U \cap V = \emptyset$, and $U \cup V = X$.

Then, if we let $A = U$, $B = V$, it follows that $A \neq \emptyset$, $B \neq \emptyset$, and $A \cup B = X$, Furthermore, $\bar{A} = A$, because A is the complement in X of the open set V, so A is closed. And then $\bar{A} \cap B = A \cap B = U \cap V = \emptyset$. Similarly, $A \cap \bar{B} = \emptyset$. So A and B satisfy conditions (D1,2,3).

To prove the converse, assume A and B satisfy (D1,2,3). Then $\bar{A} \subseteq A$, because $\bar{A} \subseteq A \cup B$ and $\bar{A} \cap B = \emptyset$. So $\bar{A} = A$, and then A is closed, so B is open. Similarly, A is open. So if we take $U = A$, $V = B$, we see that the conditions of Definition 1 hold, so X is disconnected. **Q.E.D.**

2 Connected subsets of a metric space

Now that we know what a "connected metric space" is, we also know what it means for a subset S of a metric space X to be connected. Indeed, as we have explained before, every subset of a metric space is a metric space in its own right: if X is a metric space with distance function d_X, and S is a subset of X, we can define a distance function[1] $d_S : S \times S \mapsto \mathbb{R}$ by letting

$$d_S(p, q) = d_X(p, q) \qquad \text{if} \qquad p.q \in S\,.$$

Using this, we can define:

Definition 2. Let X be a metric space with distance function d_X, and let S be a subset of X. We say that S is a connected subset of X if S, regarded as a metric space, with its distance function d_S, is a connected metric space in the sense of Definition 1. □

3 Characterization of connected subsets

In this subssection we show that the notion of "connected subset" defined here is exactly equivalent to the one given in the book, except for the fact that the book works only with subsets of the real line $\mathbb{R}$, whereas we are considering arbitrary subsets of an arbitrary metric space.

Theorem 2. *Let X be a metric space with distance function d_X, and let S be a subset of X. Then S is a connected subset of X if and only if there do not exist two subsets A, B of S such that*

(D'1) *A and B are nonempty,*

(D'2) $A \cup B = S$,

(D'3) $\bar{A} \cup B = \emptyset$ *and* $A \cup \bar{B} = \emptyset$.

Remark 1. The conditions of this theorem seem to be just restatements of Conditions (D1,2,3) of Theorem 1, applied to S instead of X. But they are not, for a somewhat subtle reason that I will now explain.

[1] In general: if A, B are sets, $f : A \mapsto B$ is a function, and C is a subset of A, then the restriction of f to C is the function $g : C \mapsto B$ given by $g(c) = f(c)$ for every $c \in C$. So the distance function d_S is the restriction of d_X to $S \times S$.

If X is a metric space, and S is a subset of X, then we can regard S as a metric space. If A is a subset of S, then we can talk about the "closure" of a subset A of S, because S is a metric space, A is a subset of S, and every subset of a metric space has a closure. Let us call this closure $\mathrm{Clos}_S(A)$. Then $\mathrm{Clos}_S(A)$ is the set of all points of S that are the limit of a sequence of points of A.

However, A is a subset of X as well, so we can also talk about the "closure" of A *in* X, meaning: the set of all points *of* X that are the limit of a sequence of points of A. Let us call this closure $\mathrm{Clos}_X(A)$.

These two concepts of "closure" are not exactly the same. For example, suppose X is $\mathbb{R}$, the real line, and S is the open interval $(0,1)$. Consider the set $A = (0, \frac{1}{2})$. Then the closure $\mathrm{Clos}_X(A)$ is the closed interval $[0, \frac{1}{2}]$, whereas the closure $\mathrm{Clos}_S(A)$ is the half-open, half-closed interval $(0, \frac{1}{2}]$. (The point 0 belongs to $\mathrm{Clos}_X(A)$, because it is a limit point of A in X, but it is not in $\mathrm{Clos}_S(A)$ because it is not in S.)

One this is understood, it should be clear that the statement of Theorem 2 is problematic: ***when the statement mentions $\bar{A}$ and $\bar{B}$, the closures of A and B, which closures is it talking about,*** $\mathrm{Clos}_X(A)$ ***and*** $\mathrm{Clos}_X(B)$***, or*** $\mathrm{Clos}_S(A)$ ***and*** $\mathrm{Clos}_S(B)$***?***

If we take "$\bar{A}$" and "$\bar{B}$" to mean "$\mathrm{Clos}_S(A)$" and "$\mathrm{Clos}_S(B)$", then Theorem 2 says nothing new: it is just the restatement of the condition for S to be a connected metric space given in Theorem 1. If, on the other hand, we take "$\bar{A}$" and "$\bar{B}$" to mean "$\mathrm{Clos}_X(A)$" and "$\mathrm{Clos}_X(B)$", then Theorem 2 makes a different assertion, and needs proof.

It turns out that the correct interpretation of Theorem 2 is the latter: "$\bar{A}$" means "$\mathrm{Clos}_X(A)$", and "$\bar{B}$" means "$\mathrm{Clos}_X(B)$". This gives us exactly the characterization of connectedness presented in the book. (As I explained before, the book does not talk about metric spaces, let alone about subsets of a metric space being themselves metric spaces. It only talks about $\mathbb{R}$ and subsets of $\mathbb{R}$, and when it mentions the closure $\bar{A}$ of a subset A of a subset S of $\mathbb{R}$, it means $\mathrm{Clos}_{\mathbf{R}}(A)$, because the book does not talk about S being a metric space in its own right.)

With this interperetation of the meaning of $\bar{A}$ and $\bar{B}$, Theorem 2 makes a truly new assertion: its Condition (D'3) says that

(D'3_X) $\mathrm{Clos}_X(A) \cap B = \emptyset$ and $A \cap \mathrm{Clos}_X(B) = \emptyset$.

This is different from the condition that occurs in the definition of "connected metric space" applied to S: if we want to regard a subset S of X

as a metric space in its own right, then the precise translation of the condition that S is disconnected has to involve the closures of A and B in S. In particular, Condition (D'3) truly says that

(D'3_S) $\mathrm{Clos}_S(A) \cap B = \emptyset$ and $A \cap \mathrm{Clos}_S(B) = \emptyset$.

Then the difference between the condition that the metric space S is connected and the condition of Theorem 2 is as follows:

(1) The metric space S (that is, the subset S of the metric space X, regarded as a metric space in its own right) is connected if and only if there do not exist sets A, B such that (D'1), (D'2), and (D'3_S) are true.

(2) The subset S of the metric space X satisfies the condition of Theorem 2 if and only if there do not exist sets A, B such that (D'1), (D'2), and (D'3_X) are true.

So, as you can see, the condition of Theorem 2 is not exactly a restatement of the fact that the metric space S is connected. It follows that *Theorem 2 needs proof.* To prove it, we have to show that Conditions (D'1), (D'2), (D'3_S) are equivalent to Conditions (D'1), (D'2), (D'3_X). This will clearly follow if we prove that

(*) If A, B satisfy (D'1) and (D'2), then

(1.1) $$\mathrm{Clos}_S(A) \cap B = \mathrm{Clos}_X(A) \cap B$$

and

(1.2) $$A \cap \mathrm{Clos}_S(B) = A \cap \mathrm{Clos}_X(B)\,.$$

Proof of Theorem 2. As explained in Remark 1, we have to prove (*).

Let us prove (1.1). Let $\mathrm{Lim}_X(A)$ be the set of all points of X that are limit points of A, and let $\mathrm{Lim}_S(A)$ be the set of all points of S that are limit points of A. Then

$$\mathrm{Clos}_X(A) = A \cup \mathrm{Lim}_X(A)\,,$$

and

$$\mathrm{Clos}_S(A) = A \cup \mathrm{Lim}_S(A)\,.$$

It is clear that $\mathrm{Lim}_S(A) \subseteq \mathrm{Lim}_X(A)$, so

$$\mathrm{Lim}_S(A) \cap B \subseteq \mathrm{Lim}_X(A) \cap B\,.$$

On the other hand, if $p \in \mathrm{Lim}_X(A) \cap B$ then $p \in S$, because $B \subseteq S$, so $p \in \mathrm{Lim}_S(A)$, and then $p \in \mathrm{Lim}_S(A) \cap B$. So

$$\mathrm{Lim}_X(A) \cap B \subseteq \mathrm{Lim}_S(A) \cap B\,.$$

Hence

$$\mathrm{Lim}_X(A) \cap B = \mathrm{Lim}_S(A) \cap B\,.$$

Therefore

$$\begin{aligned}
\mathrm{Clos}_X(A) \cap B &= (A \cup \mathrm{Lim}_X(A)) \cap B \\
&= (A \cap B) \cup (\mathrm{Lim}_X(A)) \cap B) \\
&= (A \cap B) \cup (\mathrm{Lim}_S(A)) \cap B) \\
&= (A \cup \mathrm{Lim}_S(A)) \cap B \\
&= \mathrm{Clos}_S(A) \cap B\,.
\end{aligned}$$

This proves (1.1). The proof of (1.2) is similar. **Q.E.D.**

4 Unions of connected sets

In general, the union of two connected sets need not be connected. (For example, the intervals $(0,1)$ and $(2,3)$ are connected, but their union is not connected.)

On the other, hand, if A and B are connected sets, *and* $A \cap B$ *is nonempty*, then $A \cup B$ is connected. Actually, the same result is true for an arbitrary family of connected sets.

Theorem 3. *Let X be a metric space, and let $(C_i)_{i\in I}$ be a family of connected subsets of X, such that*

$$\bigcap_{i\in I} C_i \neq \emptyset\,.$$

Let

$$C = \bigcup_{i\in I} C_i\,.$$

Then C is connected.

Proof. Let us assume that C is disconnected. Then we may write $C = A \cup B$, where $A \neq \emptyset$, $B \neq \emptyset$, $\bar{A} \cap B = A \cap \bar{B} = \emptyset$.

Pick a point $p \in \bigcap_{i\in I} C_i$. (This is possible because of our assumption that $\bigcap_{i\in I} C_i \neq \emptyset$.) Then p must belong to A or to B, so we may assume without loss of generality that $p \in A$. (If $p \in B$, then just change the names of A and B, and relabel them as B, A.)

We will prove that $C = A$. For this purpose, observe that $A \subseteq C$, because $C = A \cup B$. So we have to prove that $C \subseteq A$. In order to establish this, we show that $C_i \subseteq A$ for each $i \in I$.

So let us fix an index $i \in I$.

Let $A_* = A \cap C_i$, $B_* = B \cap C_i$. Then

$$A_* \cup B_* = (A \cap C_i) \cup (B \cap C_i) = (A \cup B) \cap C_i = C \cap C_i = C_i\,.$$

Also, $A_* \neq \emptyset$, because $p \in A_*$.

Furthermore, $\bar{A}_* \cap B_* = \emptyset$, because $\bar{A}_* \subseteq \bar{A}$ (since $A_* \subseteq A$) and $B_* \subseteq B$, so $\bar{A}_* \cap B_* \subseteq \bar{A} \cap B = \emptyset$. And, similarly, $A_* \cap \bar{B}_* = \emptyset$.

So we have shown that $A_* \neq \emptyset$, $A_* \cup B_* = C_i$, $\bar{A}_* \cap B_* = \emptyset$, and $A_* \cap \bar{B}_* = \emptyset$. If, in addition to all this, B_* was nonempty, then it would follow that C_i is disconnected, which is impossible since C_i is connected. Therefore $B_* = \emptyset$. Hence $A_* = C_i$, Therefore

$$C_i \subseteq A\,. \tag{1.3}$$

Since (1.3) is true for an arbitrary index $i \in I$, we can conclude that $C \subseteq A$.

Since $C = A \cup B$ and $A \cap B = \emptyset$, the fact that $C \subseteq A$ implies that $B = \emptyset$. But $B \neq \emptyset$, so we have reached a contradiction. This shows that C is connected, as desired. **Q.E.D.**

5 Connected subsets of the real line

The following theorem tells us that the connected subsets of $\mathbb{R}$ are easy to characterize: they are just the intervals.

Remark 2. We recall that an <u>interval</u> is a subset S of $\mathbb{R}$ such that

(*) For all real numbers a, b, c such that $a < b < c$, if a and c belong to S then it follows that b belongs to S. □

Theorem 4. *A subset S of $\mathbb{R}$ is connected if and only if S is an interval.*

Proof. The book gives a detailed proof, so I will not repeat the proof here.

Remark 3. What makes Theorem 4 very important is that it can be used in conjunction with the result of the next section—that continuous functions map connected sets to connected sets—to obtain lots of connected sets by mapping an interval (say, the interval $[0,1]$) via a continuous function. Such a map is called a "path", or "arc", and we will prove that if any two points of a set S can be connected by a path in S then S is connected. □

6 Continuous functions preserve connectedness

We recall that, if f is a function and S is a subset of the domain $\mathrm{dom}(f)$ of f, then the image of S under f is the set $f(S)$ of all values of f for all points s of S. That is,

$$f(S) \stackrel{\mathrm{def}}{=} \{f(s) : s \in S\}\,,$$

or, if you prefer,

$$f(S) \stackrel{\mathrm{def}}{=} \{x : (\exists s \in S) f(s) = x\}\,.$$

In this subsection we prove that ***continuous functions preserve connectedness***, that is, ***if you map a connected set*** S ***by a continuous function*** f ***the image set*** $f(S)$ ***is connected.***

Theorem 5. *Let* X, Y *be metric spaces with distance functions* d_X, d_Y. *Let* S *be a connected subset of* X, *and let* $f : S \mapsto Y$ *be a continuous function. Then the set* $f(S)$ *is connected.*

Proof. Suppose $f(S)$ is disconnected. Then we can write $f(S) = A \cup B$, where A, B are nonempty subsets of $f(S)$ such that $\bar{A} \cap B = A \cap \bar{B} = \emptyset$.

Let

$$\begin{aligned} A' &= \{s \in S : f(s) \in A\}\,, \\ B' &= \{s \in S : f(s) \in B\}\,. \end{aligned}$$

Then it is clear that

1. $A' \subseteq S$,
2. $B' \subseteq S$,
3. $S = A' \cup B'$ (because if $s \in S$ then $f(s) \in f(S)$, so $f(s) \in A$ or $f(s) \in B$, and then $s \in A'$ or $s \in B$),

4. $\bar{A}' \cap B' = \emptyset$ (because, if $\bar{A}' \cap B' \neq \emptyset$, then we can pick $p \in \bar{A}' \cap B'$. so $p \in B'$ and $p = \lim_{n\to\infty} x_n$, $x_n \in A'$; then $f(p) \in B$ and $f(p) = \lim_{n\to\infty} f(x_n)$, because f is continuous; since the $f(x_n)$ belong to A, it follows that $f(p) \in \bar{A}$; but $f(p) \in B$, because $p \in B'$, so $f(p) \in \bar{A} \cap B$, and then $\bar{A} \cap B \neq \emptyset$; but $\bar{A} \cap B = \emptyset$, so we got a contradiction).

5. $A' \cap \bar{B}' = \emptyset$ (by the same argument as in the previous step).

So S is disconnected. But S is connected, and we have obtained a contradiction. This proves that $f(S)$ is connected, concluding our proof. **Q.E.D.**

7 The Intermediate Value Theorem

If we combine the statements of Theorems 4 and 5, we get the following very powerful result:

Theorem 6. *Let X be a metric space, let S be a connected subset of X, and let $f : S \mapsto \mathbb{R}$ be a continuous function. Then*

(IVT1) The image $f(S)$ of S under f is an interval.

(IVT2) If p, q are points of S, and α is a real number such that

$$f(p) \leq \alpha \leq f(q)\,,$$

then there exists a point r of S such that $f(r) = \alpha$.

Proof. By Theorem 5, $f(S)$ is a connected subset of $\mathbb{R}$. By Theorem 4, $f(S)$ is an interval. This proves (IVT1).

To prove (IVT2), assume p, q, α are such that $p \in S$, $q \in S$, $\alpha \in \mathbb{R}$, and $f(p) \leq \alpha \leq f(q)$. Then the fact that $f(S)$ is an interval implies that $\alpha \in f(S)$, so there exists $r \in S$ such that $f(r) = \alpha$. **Q.E.D.**

8 Connected components

In this subsection we show that a subset S of a metric space is partitioned into "connected components", i.e., connected sets such that any connected subset of S a subset of one of the components.

Definition 3. Let X be a metric space, let S be a subset of X, and let $p \in S$.

1. The union of all the connected subsets A of S such that $p \in A$ is the connected component of p in S.

2. We write $C(p, S)$ to denote the connected component of p in S. □

Theorem 7. *Let X be a metric space, and let S be a subset of X. Then*

1. *For every $p \in S$, the connected component $C(p, S)$ is a connected subset of S. Furthermore: $C(p, S)$ is the largest connected subset of S that contains p. (This means: if A is any connected subset of S such that $p \in A$, it follows that $A \subseteq C(p, S)$.)*

2. *If $p \in S$ and $q \in S$, then the connected components $C(p, S)$, $C(q, S)$ are either disjoint or equal. (That is, either $C(p, S) \cap C(q, S) = \emptyset$ or $C(p, S) = C(q, S)$.)*

Proof. For each $p \in S$, let $\mathcal{A}(p)$ be the set of all connected subsets A of S such that $p \in A$. By Theorem 3, the union of all the sets in $\mathcal{A}(p)$ is connected, since the intersection of all these sets is nonempty, because it contains the point p. So $C(p, S)$ is connected. Furthermore, if A is any connected subset of S such that $p \in A$, then $A \in \mathcal{A}(p)$, so $A \subseteq C(p, S)$.

Finally, let p, q be points of S. We want to prove that the sets $C(p, S)$, $C(q, S)$ are either disjoint or equal. Suppose that they are not disjoint, i.e., that $C(p, S) \cap C(q, S) \neq \emptyset$. We want to prove that $C(p, S) = C(q, S)$. Since $C(p, S) \cap C(q, S) \neq \emptyset$ and $C(p, S)$ and $C(q, S)$ are connected, Theorem 3 implies that $C(p, S) \cup C(q, S)$ is connected. Furthermore, $p \in C(p, S) \cup C(q, S)$, so $C(p, S) \cup C(q, S) \in \mathcal{A}(p)$. It follows that $C(p, S) \cup C(q, S) \subseteq C(p, S)$, and then $C(q, S) \subseteq C(p, S)$. A similar argument shows that $C(p, S) \subseteq C(q, S)$. So $C(p, S) = C(q, S)$. **Q.E.D.**

Exercise 1. Recall that a binary relation on a set S is a subset of the Cartesian product $S \times S$, i.e., a set of ordered pairs of members of S. If R is a binary relation on S, we write "pRq" instead of "$(p, q) \in R$". A binary relation R on S is an equivalence relation on S if R satisfies the following three conditions:

1. R *is* ***reflexive***, that is,

$$(\forall x \in S) xRx \,.$$

2. R *is* ***symmetric***, that is,

$$(\forall x \in S)(\forall y \in S)(xRy \Longrightarrow yRx)\,.$$

3. R *is* ***transitive***, that is,

$$(\forall x \in S)(\forall y \in S)(\forall z \in S)\Big((xRy \wedge yRz) \Longrightarrow xRz\Big)\,.$$

An equivalence relation R on a set S gives rise to a partition of S into sets called "equivalence classes". (A partition of S is a set $\mathcal{P}$ such that: (1) every member of $\mathcal{P}$ is a nonempty subset of S, (2) if X, Y are members of $\mathcal{P}$ then either $X \cap Y = \emptyset$ or $X = Y$, (3) the union of all the members of $\mathcal{P}$ is S. The partition $\mathcal{P}$ arising from the equivalence relation R is defined as follows: for any member p of S, let $[p]_R$ be the set $\{q \in S : pRq\}$. Any set X such that $X = [p]_R$ for some $p \in S$ is called an equivalence class of R. Then $\mathcal{P}$ consists of all the equivalence classes of R.)

If S is a subset of a metric space X, define a binary relation R_c on S by letting

$$R_c = \{\,(x,y) \in S \times S : (\exists A)\Big(A \subseteq S \wedge A \text{ is connected} \wedge x \in A \wedge y \in A\Big)\,\}\,.$$

Prove that

1. R_c is an equivalence relation on S,

2. The equivalence classes of R_c are the connected components of S. □

9 Path-connected sets

We now present a method for proving connectedness of many sets. The idea is that we can prove that a set S is connected by proving that any two points of S can be joined by a path in S.

Definition 4. Let X be a metric space with distance d_X. Then

1. A path (or arc) in X is a continuous function $\gamma : [0,1] \mapsto X$.

2. If $S \subseteq X$, then a path γ is a path in S if $\gamma(t) \in S$ for all $t \in [0,1]$.

3. The <u>locus</u>, or <u>carrier</u>, of a path γ is the set $|\gamma|$ of all points $\gamma(t)$, for all $t \in [0,1]$. (So γ is a path in S if and only if $|\gamma| \subseteq S$.)
4. If $p, q \in X$, then a path γ in X <u>goes from p to q</u>, or <u>connects p to q</u>, if $\gamma(0) = p$ and $\gamma(1) = q$.
5. If $S \subseteq X$, then S is <u>path-connected</u> if for every pair p, q of points of S there exists a path in S that connects p to q. □

Theorem 8. *Let X be a metric space with distance d_X. Then every path-connected subset of X is connected.*

Proof. Let S be a path-connected subset of X.

Then either S is empty or S is nonempty.

Suppose S is empty. Then S is connected. (See Exercise 2 below.)

Now suppose S is not empty. Pick a point p of S. Let $\mathcal{A}$ be the set of all conected subsets of S that contain the point p. Let A be the union of all the members of $\mathcal{A}$. Then by Theorem 3 A is connected. Since every member of is a subset of S, it follows that $A \subseteq S$.

We now show that $S \subseteq A$. Let q be an arbitrary point of S. Since S is path-connected, and the points p, q are in S, there exists a path γ in S that connects p to q. By Theorem 4 the interval $[0,1]$ is connected. Then Theorem 5 implies that the carrier $|\gamma|$ is a connected subset of X, But $|\gamma| \subseteq S$, because γ is a path in S. So $|\gamma|$ is a connected subset of S that contaiins p. It follows that $|\gamma| \in \mathcal{A}$. So $|\gamma| \subseteq A$. Furthermore, $q \in |\gamma|$. So $q \in A$. Since q was an arbitrary point of S, we have proved that $(\forall q)(q \in S \Longrightarrow q \in A)$. That is, $S \subseteq A$.

So we have proved that $A \subseteq S$ and $S \subseteq A$. Hence $S = A$. Since A is connected, it follows that S is connected. **Q.E.D.**

Exercise 2. ***Prove*** that if X is a metric space, then the empty set is a connected subset of X.

10 The connected components of an open set

Theorem 9. *Let d be a natural number, and let U be an open subset of $\mathbb{R}^d$. Then the connected components of U are open sets.*

Proof. Let C be a connected component of U. We want to show that C is an open subset of $\mathbb{R}^d$. For this purpose, we show that if $p \in C$ is arbitrary, then tere exists a positive real number ε such that the neighborhood $V_\varepsilon(p)$ is a subset of C.

Since U is open, there exists a positive ε such that $V_\varepsilon(p) \subseteq C$. In addition, the set $V_\varepsilon(p)$ is connected. (Reason: Any point $q \in V_\varepsilon(p)$ can be joined by a path to the center p of $V_\varepsilon(p)$. So $_\varepsilon(p)$ is path-connected.) It follows that $V_\varepsilon(p) \subseteq C$, as desired. **Q.E.D.**

11 The structure of open subsets of $\mathbb{R}$

What is a general open subset of $\mathbb{R}$ like?

Theorem 10. *Let U be an open subset of $\mathbb{R}$. Then U is the union of a finite or countably infinite set $\mathcal{U}$ of pairwise disjoint open intervals.*

Proof. Let $\mathcal{U}$ be the set of all connected components of U. Then the members of $\mathcal{U}$ are connected subsets of $\mathbb{R}$, so they are intervals. Furthermore, they are open sets by Theorem 9. So they are open intervals.

Finally, the set $\mathcal{U}$ is finite or countably infinite, for the following reason: for every $I \in \mathcal{U}$ we can choose a rational number $f(I)$ belonging to I. Then f is a function from $\mathcal{U}$ into the set $\mathbb{Q}$ of all rational numbers. The function f is one-to-one because, if $I_1 \in \mathcal{U}$, $I_2 \in \mathcal{U}$, and $I_1 \neq I_2$, then $f(I_1) \neq f(I_2)$, because $f(I_1) \in I_1$, $f(I_2) \in I_2$, and $I_1 \cap I_2 = \emptyset$.

So f is a one-to-one function from $\mathcal{U}$ into the countable set $\mathbb{Q}$, and this implies that $\mathcal{U}$ is finite or countable. **Q.E.D.**

Exercise 3. ***Prove*** that it is not true that every closed subset of $\mathbb{R}$ is a union of a finite or countable colection of closed intervals. □

2 Homework assignment No. 12, due on Friday, December 4

The following is a list of recommended problems. Several of the ones marked "To Hand In" are challenging, so do as many as you can but don't be too upset if you cannot do some of them.

PROBLEM 1. Exercise 1 in Subsection 8.

PROBLEM 2. Exercise 2 in Subsection 9.

PROBLEM 3. Exercise 3 in Subsection 11.

PROBLEM 4. (TO HAND IN) Let X be a metric space with distance d_X, and let S be a connected subset of X. Prove that $\bar{S}$ is connected.

(Recall that the closure of a subset S of X is the set $\bar{S}$ of all points p of X such that $p = \lim_{n\to\infty} x_n$ for some sequence $(x_n)_{n=1}^{\infty}$ of points of S.)

PROBLEM 5. (TO HAND IN) Let $d \in \mathbb{N}$ be such that $d > 1$. For two real numbers r, R such that $0 < r < R$, define $A_d(r, R)$ to be "the set of points $p \in \mathbb{R}^d$ that are outside the ball centered at the origin of radius r, and inside the ball of radius R". Precisely,

$$A_d(r, R) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d : r < \|x\| < R\}.$$

Prove that $A_d(r, R)$ is connected. (HINT: Prove that $A_d(r, R)$ is path-connected. *If at some point you want to say that two points p, q can be connected by a path in $A_d(r, R)$, don't just say it. Either write down an explicit formula for the path, or prove rigorously that the path exists.*)

PROBLEM 6. (TO HAND IN) Let S be the unit circle in $\mathbb{R}^2$, that is, the set given by

$$S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

Prove that S is connected. (You may use the sine and cosine functions, if you like, and are allowed to use the fact that they are continuous, but if you are careful then you can manage to prove the result without using those functions. It suffices to use polynomials and the square-root function.)

PROBLEM 7. (TO HAND IN) Let G be the set of all points of $\mathbb{R}^2$ that are of the form $(x, \sin\frac{1}{x})$, for $x \in \mathbb{R}$, $x > 0$. That is,

$$G = \left\{ \left(x, \sin\frac{1}{x}\right) : x \in \mathbb{R} \wedge x > 0 \right\},$$

or, if you prefer,

$$G = \{\, p \in \mathbb{R}^2 : (\exists x \in \mathbb{R})(x > 0 \wedge p = (x, \sin\frac{1}{x}) \,\}. \tag{2.4}$$

So G is the graph of the function f, with domain $(0, +\infty)$, given by

$$f(x) = \sin\frac{1}{x} \quad \text{for} \quad x \in \mathbb{R}\,,\ x > 0\,. \tag{2.5}$$

Let σ be the vertical segment in $\mathbb{R}^2$ given by

$$\sigma = \{(0, y) : -1 \le y \le 1\}\,.$$

(That is, σ is the vertical segment joining the points $(0, -1)$ and $(0, 1)$.)

Let

$$S = G \cup \sigma\,.$$

1. ***Sketch*** a picture of the set S.
2. ***Prove*** that G is path-connected.
3. ***Prove*** that the closure of G is S.
4. ***Conclude*** that S is connected.
5. ***Prove*** that S is not path-connected.
6. ***Conclude*** that it is not true in general that the closure of a path-connected set is path-connected.

Remark 4. You may recall from Math 300 that a function is a set f having the following two properties: (1) f is a set of ordered pairs (that is, $(\forall x)(x \in f \Longrightarrow (\exists u, v)x = (u, v))$) and (2) whenever two paits (u, v), (u, w) belong to f, it follows that $v = w$. (In other words, a "function" is a set of "input-output pairs", having the property that for every input the function produces only one output.) The domain of a function f is the set $\text{dom}(f)$ of all possible inputs, that is,

$$\text{dom}(f) = \{x : (\exists y)(x, y) \in f)\,.$$

The value of a function f at a point x belonging to $\text{dom}(f)$ is the unique y such that $(x, y) \in f$. And we use $f(x)$ to denote the value of f at x. (In other words: the domain of f is the set of all objects x for which f produces an output. And, for each x in the domain, the value $f(x)$ is the output produced by f for x.) The graph of a function f is the set of all input-output pairs. This means that, for us, ***the graph of a function is the function***. In particular, the graph G defined by (2.4) and the function f defined by (2.5) are one and the same thing, so $G = f$. □

PROBLEM 8. Book, Exercise 4.3.3, page 126, Part (a).

PROBLEM 9. Book, Exercise 4.3.6, page 127.

PROBLEM 10. Book, Exercise 4.3.10, page 128.

PROBLEM 11. (TO HAND IN) Let X, Y be metric spaces with distance functions d_X, d_Y. Let c be a positive real number. A function $f : X \mapsto Y$ Lipschitz with constant c if

$$d_Y(f(p), f(q)) \leq c \cdot d_X(p, q)$$

for all $p, q \in X$.

A function $f : X \mapsto Y$ is Lipschitz if it is Lipschitz with constant c for some positive real number c.

Prove that a Lipschitz function is continuous.

PROBLEM 12. (TO HAND IN) Let X be a metric space with distance function d_X, and let S be a nonempty subset of X. Define a function $\mathrm{dist}_S : X \mapsto \mathbb{R}$ by letting

$$\mathrm{dist}_S(p) = \inf\{d_X(p, s) : s \in S\}\,.$$

If $f : X \mapsto \mathbb{R}$ is a function, the set of zeros of f is the set $Z(f)$ given by

$$Z(f) = \{p \in X : f(p) = 0\}\,.$$

1. ***Prove*** that if S is any nonemtpy subset of X, then the function dist_S is continous. (HINT: Prove that dist_S is Lipschitz with constant 1.)

2. ***Prove*** that the set of zeros of dist_S is exactly the closure $\bar{S}$ of S (that is, $Z(\mathrm{dist}_S) = \bar{S}$).

3. ***Prove*** that if $f : X \mapsto \mathbb{R}$ is a continous function, then $Z(f)$ is closed.

4. ***Conclude*** from the above that a subset S of X is the zero set of some continuous function from X to $\mathbb{R}$ if and only if S is closed. (That is: $(\forall S)\Big(S \subseteq X \Longrightarrow (\exists f)\Big((f : X \mapsto \mathbb{R} \wedge f \text{ is continuous} \wedge Z(f) = S) \Longleftrightarrow \bar{S} = S\Big)\Big)$.)

PROBLEM 13. (TO HAND IN) Book, Exercise 4.3.13, pages 128, 129.

PROBLEM 14. (TO HAND IN) Book, Exercise 4.3.14, page 129. (NOTE: My personal opinion is that it would be better to do Part (b) first, so I advise you to do that. But it may be that the author has a different approach in mind, which involves doing Part (a) first, so if you figure out how to do it that way, it's O.K. with me. Also, may I point out that when the book says "the function of Exercise 4.3.12 may be useful:, that function is exactly the one of Problem 12 above; the only difference is that Problem 12 asks you to do the same thing as Exercise 4.3.12, in the broader setting of general metric spaces.)