MATHEMATICS 502 — SPRING 2020

Theory of functions of a real variable II

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In this note,

- *K is a compact Hausdorff topological space,*
- *$C^0(K)$ is the space of comtinuous real-valued functions on K,*
- *λ is a positive linear functional on $C^0(K)$.*

Our goal is to construct a Borel measure μ on K such that

$$\lambda(f) = \int_K f\,d\mu \ \text{ for all } \ f \in C^0(K)\,. \tag{1}$$

The content of a compact set. If C is a compact subset of K, we define the content[1] of C to be the nonnegative real number $\kappa(C)$ given by

$$\kappa(C) = \inf\Big\{\,\lambda(f) : f \in C^0(K) \wedge f \geq \chi_C\,\Big\}\,. \tag{2}$$

Lemma 1.

1. *$\kappa(\emptyset) = 0$,*
2. *$\kappa(K) = \lambda(1)$,*
3. *(the monotonicity property) if C_1, C_2 are compact subsets of K and $C_1 \subseteq C_2$ then $\kappa(C_1) \leq \kappa(C_2)$,*
4. *(the subadditivity property) if C_1, C_2 are compact subsets of K then $\kappa(C_1 \cup C_2) \leq \kappa(C_1) + \kappa(C_2)$,*
5. *(the additivity property) if C_1, C_2 are disjoint compact subsets of K then $\kappa(C_1 \cup C_2) = \kappa(C_1) + \kappa(C_2)$.*

Proof. Statements 1, 2, and 3 are trivial.

To prove 4, we let f_1, f_2 be members of $C^0(K)$ such that $f_1 \geq \chi_{C_1}$ and $f_2 \geq \chi_{C_2}$. Then $f_1 + f_2 \geq \chi_{C_1 \cup C_2}$, so

$$\lambda(f_1 + f_2) \geq \kappa(C_1 \cup C_2)\,.$$

[1] Strictly speaking, $\kappa(C)$ should have been called the λ-*content* of C, and we should have named it "$\kappa_\lambda(C)$" rather than "$\kappa(C)$". But as long as λ is fixed, no harm is done.

On the other hand, $\lambda(f_1+f_2)=\lambda(f_1)+\lambda(f_2)$, so

$$\lambda(f_1)+\lambda(f_2)\geq\kappa(C_1\cup C_2)\,.$$

Taking the infimum ith respct to f_1 and then the infimum with respect to f_2, we get

$$\kappa(C_1)+\kappa(C_2)\geq\kappa(C_1\cup C_2)\,.$$

To prove 5 we use Urysohn's Lemma and pick a function $\varphi\in C^0(K)$ such that $0\leq\varphi\leq 1$, $\varphi=1$ on C_1 and $\varphi=0$ on C_2. If $\psi=1-\varphi$, then $\psi\in C^0(K)$, $0\leq\psi\leq 1$, $\psi=0$ on C_1 and $\psi=1$ on C_2. Fix a positive real number ε, and let $f\in C^0(K)$ be such that $f\geq\chi_{C_1\cup C_2}$ and $\lambda(f)\leq\kappa(C_1\cup C_2)+\varepsilon$. It follows that $\varphi f\geq\chi_{C_1}$ and $\psi f\geq\chi_{C_2}$, so $\lambda(\varphi f)+\lambda(\psi f)\geq\kappa(C_1)+\kappa(C_2)$. Therefore

$$\begin{aligned}\kappa(C_1)+\kappa(C_2) &= \lambda(\varphi f)+\lambda(\psi f)\\ &= \lambda(\varphi f+\psi f)\\ &= \lambda\big((\varphi+\psi)f\big)\\ &= \lambda(f)\\ &\leq \kappa(C_1\cup C_2)+\varepsilon\,.\end{aligned}$$

Since ε is arbitrary, it follows that

$$\kappa(C_1)+\kappa(C_2)\leq\kappa(C_1\cup C_2)\,.$$

On the other hand, the subadditivity property implies that

$$\kappa(C_1)+\kappa(C_2)\geq\kappa(C_1\cup C_2)\,.$$

Therefore

$$\kappa(C_1)+\kappa(C_2)=\kappa(C_1\cup C_2)\,,$$

completing our proof. **Q.E.D.**

The outer measure of an open set. If U is an open subset of K, we define[2] the outer measure of U to be the nonnegative real number $\mu^*(U)$ given by

$$\mu^*(U)=\sup\Big\{\kappa(C):C\subseteq U\wedge C\text{ compact}\Big\}\,. \qquad (3)$$

Lemma 2.

[2] Again, $\mu^*(U)$ should really have been called the λ-*outer measure* of U, and we should have named it "$\mu_{o,\lambda})(U)$" rather than "$\mu^*(U)$". But as long as λ is fixed, no harm is done.

1. *if C is compact, U is open, and $C \subseteq U$, then*

$$\kappa(C) \leq \mu^*(C) \leq \mu^*(U);$$

2. $\mu^*(\emptyset) = 0$,

3. $\mu^*(K) = \lambda(1)$,

4. *(the monotonicity property) if U_1, U_2 are open subsets of K and $U_1 \subseteq U_2$ then $\mu^*(U_1) \leq \mu^*(U_2)$,*

5. *(the countable subadditivity property) if $\mathbf{U} = (U_j)_{j\in\mathbf{N}}$ is a sequence of open subsets of K and $U = \cup_{j\in\mathbf{N}} U_j$, then*

$$\mu^*(U) \leq \sum_{j\in\mathbf{N}} \mu^*(U_j);$$

6. *(the countable additivity property) if $\mathbf{U} = (U_j)_{j\in\mathbf{N}}$ is a sequence of pairwise disjoint open subsets of K, and $U = \cup_{j\in\mathbf{N}} U_j$, then*

$$\mu^*(U) = \sum_{j\in\mathbf{N}} \mu^*(U_j).$$

Proof. Statements 1, 2, 3, and 4 are trivial.

To prove 5, we fix a positive real number ε, and let C be a compact subset of U such that $\kappa(C) + \varepsilon \geq \mu^*(U)$, so

$$\mu^*(U) - \varepsilon \leq \kappa(C).$$

Since the sets U_j, for $j \in \mathbf{N}$, form an open covering of C, we can pick $N \in \mathbf{N}$ such that $C \subseteq \cup_{j=1}^N U_j$. Then we can find[3] compact subsets $C_1, C_2, \ldots, C_N$ of $U_1, U_2, \ldots, U_N$ such that $C = \cup_{j=1}^N C_j$. By the subadditivity property of the content,

$$\kappa(C) \leq \sum_{j=1}^N \kappa(C_j).$$

[3] *Proof*: For each $x \in C$, let $j(x)$ be the smallest member j of the set $\{1, 2, \ldots, N\}$ such that $x \in U_j$. Then let V_x be a compact neighborhood of x such that $V_x \subseteq U_{j(x)}$. Then let X be a finite subset of C such that the sets V_x, $x \in X$, cover C. For each $j \in \{1, 2, \ldots, N\}$, let $D_j = \cup_{x\in X, \wedge j(x)=j} V_x$. Let $D = \cup_{j=1}^N D_j$. Then each D_j is compact, so D is compact. Furthermore, $C \subseteq D$. In addition, $D_j \subseteq U_j$ for each j, So, if we let $C_j = D_j \cap C$, it follows that the C_j are compact, $C_j \subseteq U_j$, and $C = \cup_{j=1}^N C_j$.

Then

$$\begin{aligned}\mu^*(U)-\varepsilon &\leq \kappa(C)\\ &\leq \sum_{j=1}^{N}\kappa(C_j)\\ &\leq \sum_{j=1}^{\infty}\kappa(C_j)\\ &\leq \sum_{j=1}^{\infty}\mu^*(U_j)\,,\end{aligned}$$

so

$$\mu^*(U)-\varepsilon \leq \sum_{j=1}^{\infty}\mu^*(U_j)\,.$$

Since ε is arbitrary, it follows that

$$\mu^*(U) \leq \sum_{j=1}^{\infty}\mu^*(U_j)\,,$$

proving the countable subadditivity property.

To prove additivity, we consider first the case of two disjoint open sets. Let U_1, U_2 be open and such that $U_1 \cap U_2 = \emptyset$. Let C_1, C_2 be arbitrary compact subsets of U_1, U_2. Then $C_1 \cap C_2 = \emptyset$, so

$$\kappa(C_1 \cup C_2) = \kappa(C_1) + \kappa(C_2)\,.$$

It follows that

$$\kappa(C_1) + \kappa(C_2) \leq \mu^*(U_1 \cup U_2)\,, \tag{4}$$

since $C_1 \cup C_2$ is a compact subset of $U_1 \cup U_2$.

Taking the supremum over all compact subsets C_1 of U_1, and then the supremum over all compact subsets C_2 of U_2, we find

$$\mu^*(U_1) + \mu^*(U_2) \leq \mu^*(U_1 \cup U_2)\,. \tag{5}$$

Since $\mu^*(U_1) + \mu^*(U_2) \geq \mu^*(U_1 \cup U_2)$ by the subaditivity property, we end up with

$$\mu^*(U_1) + \mu^*(U_2) = \mu^*(U_1 \cup U_2)\,. \tag{6}$$

From this it follows easily by induction that if $U_1, \ldots, U_N$ are pairwise disjoint open sets, then

$$\mu^*\Big(\bigcup_{j=1}^{N} U_j\Big) = \sum_{j=1}^{N} \mu^*(U_j)\,. \tag{7}$$

Finally, if $\mathbf{U} = (U_j)_{j\in\mathbf{N}}$ is a sequence of pairwise disjoint open subsets of K, and $U = \cup_{j\in\mathbf{N}} U_j$, then

$$\begin{aligned} \mu^*(U) &\geq \mu^*\Big(\bigcup_{j=1}^{N} U_j\Big) \\ &= \sum_{j=1}^{N} \mu^*(U_j) \end{aligned}$$

for every N, so

$$\mu^*(U) \geq \sum_{j=1}^{\infty} \mu^*(U_j)\,.$$

Since we know from the countable subadditivity property that

$$\mu^*(U) \leq \sum_{j=1}^{\infty} \mu^*(U_j)\,,$$

it follows that

$$\mu^*(U) = \sum_{j=1}^{\infty} \mu^*(U_j)\,,$$

completing our proof.

The outer measure of an arbitrary set. If E is an arbitrary subset of K, we define[4] the <u>outer measure</u> of E to be the nonnegative real number $\mu^*(E)$ given by

$$\mu^*(E) = \inf\Big\{\,\mu^*(U) : E \subseteq U \subseteq K \wedge U \text{ compact}\Big\}\,. \tag{8}$$

Lemma 3. *IF E is open then the number $\mu^*(E)$ defined by Equation (8) agrees with the number $\mu^*(E)$ defined by Equation (3).*

[4] Strictly speaking, we cannot call this new quantity "$\mu^*(E)$" for arbitrary E, because when E is an open set we have already defined what "$\mu^*(E)$" is. But it is completely obvious (and we will prove it soon, in Lemma 3) that for an open set E the new $\mu^*(E)$ agrees with the old one, so no harm is done.

Proof. Temporarily, let us use "$\mu^{*,new}(E)$" for the right-hand side of Equation (8). Then it is clear that if E is open then $\mu^{*,new}(E) = \mu^*(E)$, because $\mu^*(E) \leq \mu^*(U)$ for every open set U such that $E \subseteq U$, and one of those sets is E itself. This completes the proof. **Q.E.D.**

Lemma 4.

1. *if E is an arbitrary subset of K, U is open, C is compact, and $C \subseteq E \subseteq U$, then*
$$\kappa(C) \leq \mu^*(C) \leq \mu^*(E) \leq \mu^*(U)\,.$$
2. $\mu^*(\emptyset) = 0$,
3. $\mu^*(K) = \lambda(1)$,
4. *(the monotonicity property) if E_1, E_2 are subsets of K and $E_1 \subseteq E_2$ then $\mu^*(E_1) \leq \mu^*(E_2)$,*
5. *(the countable subadditivity property) if $\mathbf{E} = (E_j)_{j\in\mathbf{N}}$ is a sequence of open subsets of K and $E = \cup_{j\in\mathbf{N}} E_j$, then*
$$\mu^*(E) \leq \sum_{j\in\mathbf{N}} \mu^*(E_j)\,.$$

Proof. Statements 1, 2, 3, and 4 are trivial.

To prove 5, we fix a positive real number ε, and let U_j be open subsets of K such that $E_j \subseteq U_j$ and
$$\mu^*(U_j) \leq \mu^*(E_j) + 2^{-j}\varepsilon\,.$$

Let $U = \cup_{j=1}^{\infty} U_j$, so U is open and $E \subseteq U$. Then

$$\begin{aligned}
\mu^*(E) &\le \mu^*(U) \\
&= \mu^*\Big(\bigcup_{j=1}^{\infty} U_j\Big) \\
&\le \sum_{j=1}^{\infty} \mu^*(U_j) \\
&\le \sum_{j=1}^{\infty} \mu^*(U_j) \\
&\le \sum_{j=1}^{\infty} \Big(\mu^*(E_j) + 2^{-j}\varepsilon\Big) \\
&= \sum_{j=1}^{\infty} \mu^*(E_j) + \sum_{j=1}^{\infty} 2^{-j}\varepsilon \\
&= \sum_{j=1}^{\infty} \mu^*(E_j) + \varepsilon\,.
\end{aligned}$$

So

$$\mu^*(E) \le \sum_{j=1}^{\infty} \mu^*(E_j) + \varepsilon$$

and, since ε is arbitrary, we find that

$$\mu^*(E) \le \sum_{j=1}^{\infty} \mu^*(E_j)\,,$$

completing our proof. **Q.E.D.**

The outer measure of a compact set.

Theorem 1. *If C is a compact subset of K, then $\mu^*(C) = \kappa(C)$.*

Proof. We know that $\kappa(C) \le \mu^*(C)$, so all we need is to prove that

$$\mu^*(C) \le \kappa(C)\,. \tag{9}$$

Let $\alpha = \kappa(C)$. Pick a positive real number ε, and a function $f \in C^0(K)$ such that $f \ge \chi_C$ and $\lambda(f) \le \alpha + \varepsilon$.

Fix a real number θ such that $0 < \theta < 1$. Let $V_\theta = \{x \in K : f(x) > \theta\}$. Then V_θ is open and $C \subseteq V_\theta$. Then the function $K \ni x \mapsto g_\theta(x) \stackrel{\text{def}}{=} \frac{f(x)}{\theta}$ is ≥ 1 on V_θ, so $\mu^*(V_\theta) \leq \lambda(g_\theta)$ (because

$$\mu^*(V_\theta) = \sup\{\,\kappa(D) : D \subseteq V_\theta \wedge D \text{ compact}\,\}$$

and if D is an arbitrary compact subset of V_θ then $g_\theta \geq \chi_D$, so $\kappa(D) \leq \lambda(g_\theta)$).

Clearly,

$$\lambda(g_\theta) = \frac{\lambda(f)}{\theta} \leq \frac{\alpha + \varepsilon}{\theta}\,,$$

so

$$\mu^*(V_\theta) \leq \frac{\alpha + \varepsilon}{\theta}\,.$$

Since $C \subseteq V_\theta$, it follows that $\mu^*(C) \leq \mu^*(V_\theta)$, so

$$\mu^*(C) \leq \frac{\alpha + \varepsilon}{\theta}\,.$$

Since θ and ε are arbitrary (in the ranges $(0, 1)$ and $(0, \infty)$, respectively, we conclude that $\mu^*(C) \leq \alpha$, so (9) holds. **Q.E.D.**

Measurable sets. Recall that, in general,

- An <u>outer measure</u> on a set X is a function[5] $\nu : 2^X \mapsto [0, +\infty]$ such that $\nu(\emptyset) = 0$ and ν satisfies the monotonicity and countable subadditivity properties
- An outer measure ν on a set X is <u>finite</u> if[6] $\mu(X) < \infty$.

It follows that

Corollary 1. *The function μ^* that we have constructed is a finite outer measure on K.*

Next, we recall the following general procedure, due to Carathéodory, for constructing a measure from an outer measure ν on a set X:

1. Call two subsets A, B of X <u>nicely dsjoint</u>[7] if $A \cap B = \emptyset$ and $\nu(A \cup B) = \nu(A) + \nu(B)$.

[5] "2^X" is the <u>power set</u> of X, i.e., the set of all subsets of X. And, naturally, "$[0, +\infty]$" is the <u>nonnegative extended real line</u>, i.e., the union $\{x \in \mathbf{R} : \mathbf{x} \geq \mathbf{0}\} \cup \{+\infty\}$.

[6] Obviously, ν is finite if and only if $\mu(S)$ is finite for every subset S of X.

[7] Of course, we should have said "nicely disjoint with respecto ν", but as long as ν is fixed what we are doing is O.K.

2. IF $S \subseteq X$, call S $\underline{\nu\text{-measurable}}$ if for every subset E of X the sets[8] $S \cap E$, $S^c \cap E$ are nicely disjoint.

We use $\mathcal{M}(\nu)$ to denote the set of all ν-measurable subsets of X. Then the following is the key theorem on the Carathéodory construction:

Theorem 2. *Let ν be an outer measure on a set X. Then*

1. *$\mathcal{M}(\nu)$ is a σ-algebra of subsets of X,*
2. *the restriction $\nu\lceil\mathcal{M}(\nu)$ of ν to $\mathcal{M}(\nu)$ is a measure.*

It follows from Theorem 2 and Corollary 1 that the μ^*-measurable subsets of K form a σ-algebra*and*the restriction of μ^* to this σ-algebra is a measure. The σ-algebra ought to be called $\mathcal{M}(\mu^*)$ but, since μ^* was constructed from the functional λ, we will call it $\mathcal{M}(\lambda)$ instead. And the measure obtained by restricting μ^* to $\mathcal{M}(\lambda)$ will be called μ_λ.

Hence we have defined, for each compact Hausdorff space X and each positive linear functional λ on $C^0(K)$, a σ-algebra $\mathcal{M}(\lambda)$ of subsets of K and a finite measure $\mu_\lambda : \mathcal{M}(\lambda) \mapsto [0, +\infty)$.

Remark 1. ***For a given compact Hausdorff space K, the σ-algebra $\mathcal{M}(\lambda)$ in general depends on λ***, as shown by the following two examples.

Example 1. (Lebesgue measure) Let $K = [0, 1]$. Let $\lambda(f) = \int_0^1 f(x)dx$, where the integral is a Riemann integral[9].

Then it is easy to see that the σ-algebra $\mathcal{M}(\lambda)$ corresponding to λ is the set of all Lebesgue-measurable subsets of $[0, 1]$ and the measure μ_λ is Lebesgue measure. □

Example 2. (The Dirac delta functions) Let $K = [0, 1]$. Fix a point $p \in [0, 1]$., and define

$$\lambda_p(f) = f(p) \text{ for } f \in C^0(K)\,.$$

Then it is easy to verify that the σ-algebra $\mathcal{M}(\lambda_p)$ corresponding to λ_p is the set of all subsets of $[0, 1]$ and the measure μ_{λ_p} is the "Dirac delta function at p, given by

$$\mu_{\lambda_p}(S) = \begin{cases} 0 & \text{if} \quad p \notin S \\ 1 & \text{if} \quad p \in S \end{cases}.$$

□

[8] We use "S^c" for the complement of S relative to X, so $S^c = \{x \in X : x \notin S\}$.

[9] It is well known that, on a compact interval $[a, b]$, every continuous function is Riemann-integrable and, furthermore, the integral of a nonnegative functiom is a nonnegative real number.

Measurability of Borel sets.

Theorem 3. *If K is a compact Hausdorff space and λ is a positive lieanr functional on $C^0(K)$, then every Borel set is μ_λ-measurable.*

Proof. We know that the set $\mathcal{M}(\lambda)$ of all μ_λ-measurable sets is a σ-algebra. So, in order to prove that every Borel subset of K is μ_λ-measurable, it suffices to prove that every open subset of K is μ_λ-measurable.

Let U be an open subset of K. To prove that U is measurable we have to prove that if E is an arbitrary suhset of K, then

$$\mu^*(E \cap U) + \mu^*(E \cap U^c) = \mu^*(E)\,. \tag{10}$$

Furtherrmore, the subadditivity of μ^* implies the inequality

$$\mu^*(E \cap U) + \mu^*(E \cap U^c) \geq \mu^*(E)\,, \tag{11}$$

so all we need is to prove that

$$\mu^*(E \cap U) + \mu^*(E \cap U^c) \leq \mu^*(E)\,. \tag{12}$$

Fix a positive real number ε, and pick an open subset V of K such that

$$E \subseteq V \ \text{ and } \ \mu^*(E) + \varepsilon \geq \mu^*(V)\,. \tag{13}$$

Next, pick open sets $\tilde{V}_1$, $\tilde{V}_2$, such that

$$E \cap U \subseteq \tilde{V}_1 \ \text{ and } \ E \cap U^c \subseteq \tilde{V}_2\,. \tag{14}$$

Let $V_1 = V \cap \tilde{V}_1$, $V_2 = V \cap \tilde{V}_2$. Then

$$E \cap U \subseteq V_1 \ \text{ and } \ E \cap U^c \subseteq V_2\,. \tag{15}$$

Let $W_1 = U \cap V_1$, so

$$W_1 \ \text{ is open and } \ E \cap U \subseteq W_1\,. \tag{16}$$

Then pick a compact subset C_1 of K such that

$$C_1 \subseteq W_1 \ \text{ and } \ \kappa(C_1) + \varepsilon \geq \mu^*(W_1)\,. \tag{17}$$

It then follows, since $\mu^*(W_1) \geq \mu **(E \cap U)$ (because $E \cap U \subseteq W_1$), that

$$C_1 \subseteq U \ \text{ and } \ \kappa(C_1) + \varepsilon \geq \mu^*(W_1) \geq \mu^*(E \cap U)\,. \tag{18}$$

Let $W_2 = V_2 \cap C_1^c$. Then W_2 is open, because V_2 and C_1^c are open. Furthermore,

$$E \cap U^c \subseteq W_2\,, \tag{19}$$

because (a) $E \cap U^c \subseteq V_2$ and (b) $E \cap U^c \subseteq C_1^c$, because $C_1 \subseteq U$.

Now pick a compact subset C_2 of K such that

$$C_2 \subseteq W_2 \text{ and } \kappa(C_2) + \varepsilon \geq \mu^*(W_2)\,. \tag{20}$$

It then follows, since $\mu^*(W_2) \geq \mu^*(E \cap U^c)$ (because $E \cap U^c \subseteq W_2$) and $C_2 \subseteq W_2 \subseteq C_1^c$, that

$$C_1 \cap C_2 = \emptyset \text{ and } \kappa(C_2) + \varepsilon \geq \mu^*(W_2) \geq \mu^*(E \cap U^c)\,. \tag{21}$$

Then (18) and (21), together with the additivity property for the content, that

$$\begin{aligned}
\kappa(C_1 \cup C_2) &= \kappa(C_1) + \kappa(C_2) \\
&\geq \big(\mu^*(W_1) - \varepsilon\big) + \big(\mu^*(W_2) - \varepsilon\big) \\
&= \mu^*(W_1) + \mu^*(W_2) - 2\varepsilon \\
&\geq \mu^*(E \cap U) + \mu^*(E \cap U^c) - 2\varepsilon\,,
\end{aligned}$$

so

$$\kappa(C_1 \cup C_2) \geq \mu^*(E \cap U) + \mu^*(E \cap U^c) - 2\varepsilon\,. \tag{22}$$

On the other hand,

$$C_1 \cup C_2 \subseteq W_1 \cup W_2 \subseteq V_1 \cup V_2 \subseteq V\,. \tag{23}$$

Hence

$$\mu^*(V) \geq \kappa(C_1 \cup C_2)\,. \tag{24}$$

It then follows from (13) and (24) that

$$\mu^*(E) + \varepsilon \geq \kappa(C_1 \cup C_2)\,. \tag{25}$$

This, together with (22), imply that $\mu^*(E) + \varepsilon \geq \mu^*(E \cap U) + \mu^*(E \cap U^c) - 2\varepsilon$, so

$$\mu^*(E) + 3\varepsilon \geq \mu^*(E \cap U) + \mu^*(E \cap U^c)\,. \tag{26}$$

Since ε is arbitrary, the desired inequality

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \cap U^c)$$

follows. **Q.E.D.**

Recovering the linear functional from the measure. Having constructed the Borel measure μ_λ, we still have to prove that μ_λ has the desired property, i.e., that

Theorem 4. *For every $f \in C^0(K)$,*

$$\int_K f(x)d\mu_\lambda(x) = \lambda(f)\,. \tag{27}$$

Proof. In this proof, "μ" stands for "μ_λ".

Naturally, it suffices to prove (27) for positive f.

So let $f \in C^0(K)$, $f \geq 0$, and fix a positive real umber ε.

For each nonnegative integer k, let

$$C_{\varepsilon,k} = \{x \in K : f(x) \geq \varepsilon k\}\,. \tag{28}$$

Then $C_{\varepsilon,0} = K$, and the sets $C_{\varepsilon,k}$ are compact, decreasing, and empty for sufficientlt large k, i.e.,

$$K = C_{\varepsilon,0} \supseteq C_{\varepsilon,1} \supseteq C_{\varepsilon,2} \supseteq \cdots \supseteq C_{\varepsilon,N_1} \supseteq C_{\varepsilon,N} = \emptyset\,,$$

if $N \in \mathbf{N}$ is such that $N > \max\{f(x) : x \in K\}$.

Let

$$D_{\varepsilon,k} = C_{\varepsilon,k} - C_{\varepsilon,k+1}\,,$$

so the $D_{\varepsilon,k}$ are Borel measurable pairwise disjoint subsets of K and constitute a partition of K.

Define functions $g_{\varepsilon,k}$, for $k \in \{0, 1, \ldots, N\}$, by letting

$$g_{\varepsilon,k} = \begin{cases} 0 & \text{if} \quad f(x) \leq \varepsilon k \\ \frac{1}{\varepsilon}\big(f(x) - \varepsilon k\big) & \text{if} \quad \varepsilon k \leq f(x) \leq \varepsilon(k+1) \\ 1 & \text{if} \quad \varepsilon(k+1) \leq f(x) \end{cases}\,.$$

Then the $g_{\varepsilon,k}$ are positive, continuous, and such that

$$f = \varepsilon \sum_{k=0}^{N} g_{\varepsilon,k}\,.$$

Furthermore, each $g_{\varepsilon,k}$ satisfies

$$\begin{array}{lll} g_{\varepsilon,k} \equiv 0 & \text{on} & K - C_{\varepsilon,k} \\ 0 \leq g_{\varepsilon,k} \leq 1 & \text{on} & D_{\varepsilon,k} \\ g_{\varepsilon,k} \equiv 1 & \text{on} & C_{\varepsilon,k+1} \end{array}\,. \tag{29}$$

It follows from this that

$$\lambda(g_{\varepsilon,k}) \geq \kappa(C_{\varepsilon,k+1}) = \mu^*(C_{\varepsilon,k+1}) \tag{30}$$

and also that

$$\lambda(g_{\varepsilon,k}) \leq \mu^*(C_{\varepsilon,k}) \tag{31}$$

(because, if we let $h_{\varepsilon,k} = 1 - g_{\varepsilon,k}$, and $E_{\varepsilon,k} = K - C_{\varepsilon,k}$, then $h_{\varepsilon,k} \geq \chi_{E_{\varepsilon,k}}$, and $E_{\varepsilon,k}$ is open; since $h_{\varepsilon,k} \geq \chi_J$ for every compact subset J of $E_{\varepsilon,k}$,it follows that $\lambda(h_{\varepsilon,k}) \geq \kappa(J)$ for every such J, so $\lambda(h_{\varepsilon,k}) \geq \mu^*(E_{\varepsilon,k})$; finally, $\mu^*(E_{\varepsilon,k}) = \mu^*(K) - \mu^*(C_{\varepsilon,k})$, $\lambda(h_{\varepsilon,k}) = \lambda(1) - \lambda(g_{\varepsilon,k})$, and $\lambda(1) = \mu^*(K)$, so (31) follows).

On the other hand, the inequalities (29) clearly imply that

$$\mu^*(C_{\varepsilon,k+1}) \leq \int_K g_\varepsilon, k \, d\mu \leq \mu^*(C_{\varepsilon,k}) \,. \tag{32}$$

It follows from (30), (31), and (32) that the numbers $\lambda(g_{\varepsilon,k})$ and $\int_K g_{\varepsilon,k} \, d\mu$ both belong to the closed interval $[\mu^*(C_{\varepsilon,k+1}), \mu^*(C_{\varepsilon,k})]$.

Therefore

$$\left|\lambda(g_{\varepsilon,k}) - \int_K g_{\varepsilon,k} \, d\mu\right| \leq \mu^*(C_{\varepsilon,k}) - \mu^*(C_{\varepsilon,k+1}) = \mu^*(D_{\varepsilon,k}) \,. \tag{33}$$

Hence

$$\begin{aligned} \left|\lambda(f) - \int_K f \, d\mu\right| &= \left|\lambda\big(\varepsilon \sum_{k=0}^N g_{\varepsilon,k}\big) - \int_K \big(\varepsilon \sum_{k=0}^N g_{\varepsilon,k}\big) \, d\mu\right| \\ &\leq \varepsilon \sum_{k=0}^N \left|\lambda\big(g_{\varepsilon,k}\big) - \int_K g_{\varepsilon,k} \, d\mu\right| \\ &\leq \varepsilon \sum_{k=0}^N \mu^*(D_{\varepsilon,k}) \\ &= \varepsilon\mu^*(K) \,. \end{aligned}$$

So

$$\left|\lambda(f) - \int_K f \, d\mu\right| \leq \varepsilon\mu^*(K)$$

for arbitrary positive ε. Therefore $\lambda(f) = \int_K f \, d\mu$, and our proof is complete. **Q.E.D.**