MATHEMATICS 503 — FALL 2009

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1 Homework assignment No. 5, due on October 7

This homework assignment consists of three problems. The first one is very long and the other two are very short.

PROBLEM 1. Give a rigorous justification of the following formula

$$\pi \cot \pi z = \sum_{n\in\mathbb{Z}} \frac{1}{z-n}\,. \tag{1}$$

Here

i. By definition, $\cot z = \frac{\cos z}{\sin z}$, so $\cot z = i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$.

ii. The infinite sum on the right-hand side of (1) is interpreted as the limit $\lim_{N\to\infty} S_N(z)$, where

$$S_N(z) = \sum_{n=-N}^{N} \frac{1}{z-n}\,. \tag{2}$$

(NOTE: The "unconditional" limit $\lim_{N\to\infty,M\to\infty} \sum_{n=-M}^{N} \frac{1}{z-n}$ does not exist, because the terms of the series $\sum_{n\in\mathbb{Z}} \frac{1}{z-n}$, for a fixed noninteger z, behave like the terms of the harmonic series, which is divergent.)

iii. The convergence of the sequence $\{S_N(z)\}_{N=1}^{\infty}$ to a limit $S_\infty(z)$ is in the following sense:

(#) Given any compact subset K of $\mathbb{C}$ such that $K\cap\mathbb{Z}=\emptyset$, the functions $S_N(z)$ converge uniformly on K to $S_\infty(z)$.

(##) Given any $\nu\in\mathbb{Z}$, and any r such that $0<r<1$, the functions $S_N(z)-\frac{1}{z-\nu}$—which are analytic on the disc

$$D(\nu,r) = \{z\in\mathbb{C} : |z-\nu| < r\}\,,$$

as long as $N\geq|\nu|$—converge uniformly on $D(\nu,r)$ to the function $S_\infty - \frac{1}{z-\nu}$—which is also analytic on $D(\nu,r)$.

To prove (1),

1. First prove that the function f given by $f(z) = \pi \cot \pi z$ is meromorphic on $\mathbb{C}$, with poles at the integers. (This means that f is analytic on $\mathbb{C}\backslash\mathbb{Z}$, and has a pole at every $\nu \in \mathbb{Z}$.)

2. Show that for every integer ν the function $f(z) - \frac{1}{z-\nu}$ is analytic on a neighborhood of ν.

3. Show that f is periodic with period 1, that is,
$$f(z+1) = f(z) \qquad \text{whenever} \quad z \in \mathbb{C}\backslash\mathbb{Z}\,, \tag{3}$$
and f is odd, that is
$$f(-z) = f(z) \qquad \text{whenever} \quad z \in \mathbb{C}\backslash\mathbb{Z}\,. \tag{4}$$

4. Show that f is bounded on the complement of the strip
$$STR(r) \stackrel{\text{def}}{=} \{x + iy \in \mathbb{C} : -r < y < r\}$$
for every positive r. That is, show that whenever $r > 0$ there exists a constant C_r such that
$$|f(x+iy)| \le C_r \qquad \text{whenever} \quad |y| \ge r\,. \tag{5}$$

5. Show that
$$S_N(z) = \frac{1}{z} + \sum_{n=1}^{N} \frac{2z}{z^2 - n^2}\,, \tag{6}$$
and that $S_N(z) - \frac{1}{z-\nu}$ is analytic on $\{z : |z-\nu| < 1\}$ provided that $\nu \in \mathbb{Z}$ and $N \ge |\nu|$.

6. For z in a compact set K such that $K \cap \mathbb{Z} = \emptyset$, estimate the terms of the series
$$\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}\,, \tag{7}$$
by showing that their absolute values are bounded by $\frac{C}{n^2}$ for some constant C.

7. Conclude from the estimate of the previous step that the series converges to a limit $S_\infty(z)$ for every $z \in \mathbb{C}\backslash\mathbb{Z}$, and that the convergence is uniform on every compact set K such that $K \cap \mathbb{Z} = \emptyset$. In particular, conclude from this that the function S_∞ is analytic on $\mathbb{C}\backslash\mathbb{Z}$.

8. Use the estimate of Part 6 to show that, if $\nu \in \mathbb{Z}$ and we subtract $\frac{1}{z-\nu}$ from the series of (7) (more precisely, (a) if $\nu = 0$ then we omit the $\frac{1}{z}$ term, (b) if $\nu \neq 0$ then we subtract $\frac{1}{z-\nu}$ from the term $\frac{2z}{z^2-\nu^2}$, so $\frac{2z}{z^2-\nu^2}$ is replaced by $\frac{1}{z+\nu}$), then the resulting series converges uniformly on a neighborhood of ν to the function $S_\infty(z) - \frac{1}{z-\nu}$.

9. Prove that the function $S_\infty(z)$ is periodic of period 1 and odd. (NOTE: the periodicity of $S_\infty(z)$ is more or less obvious by the following formal argument: $S_\infty(z) = \sum_{n\in\mathbb{Z}} \frac{1}{z-n}$, so

$$S_\infty(z+1) = \sum_{n\in\mathbb{Z}} \frac{1}{z+1-n} = \sum_{n\in\mathbb{Z}} \frac{1}{z-(n-1)}\,,$$

and then, “changing variables”, and observing that as n runs through all the integers then $n-1$ also runs through all the integers, we get

$$S_\infty(z+1) = \sum_{n\in\mathbb{Z}} \frac{1}{z-n} = S_\infty(z)\,.$$

This argument, however, is not rigorous. To get a rigorous proof, you should estimate the difference $S_N(z+1) - S_N(z)$ and show that this goes to zero as $N \to \infty$.)

10. To establish the equality of $f(z) = S_\infty(z)$, here is what you can do:

    a. Prove that $f - S_\infty$ is an entire function, which is periodic of period 1 and odd.

    b. Get an estimate for S_∞ similar to the estimate obtained for f in Part 4. The easiest thing is to get a linear estimate

    $$|S_\infty(z)| \leq \text{constant} \times (1+|z|)$$

    on the complement of some strip $STR(r)$. Using the periodicity of S_∞, you only need to prove this for $0 \leq x \leq 1$.

c. Conclude from all this that $f - S_\infty$ is an entire function that satisfies a linear growth estimate, so it's a linear function $z \mapsto az + b$. Then use the oddness and periodicity to deliver the final blow and deduce that $f - S_\infty \equiv 0$.

PROBLEM 2. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} .$$

PROBLEM 3. Compute

$$\sum_{n=1}^{\infty} \frac{1}{n^4} .$$