MATHEMATICS H311 — FALL 2015

Introduction to Mathematical Analysis

H. J. Sussmann

INSTRUCTOR'S NOTES

Contents

1 Infinite sums

In this section we consider sums $\sum_{i\in I} x_i$, where

1. the index set I is a countable set[1]. For example, I could be

   (a) $\mathbb{N}$, the set of all natural numbers,

   (b) $\mathbb{N}\cup\{0\}$, the set of all nonnegative integers,

   (c) an infinite subset J of $\mathbb{N}\cup\{0\}$,

   (d) $\mathbb{N}\times\mathbb{N}$, the set of all ordered pairs (m,n) of natural numbers,

   (e) $(\mathbb{N}\cup\{0\})\times(\mathbb{N}\cup\{0\})$, the set of all ordered pairs (m,n) of nonnegative integers,

   (f) a subset of $(\mathbb{N}\cup\{0\})\times(\mathbb{N}\cup\{0\})$ of the form $J_1\times J_2$, where J_1 and J_2 are infinite subsets of $\mathbb{N}\cup\{0\}$,

[1] Why do we have to require I to be countable? This will be explained later, in Remark 2, and I will also explain what happens if I is not countable.

(g) $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, or $(\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$, or a subset of $(\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$ of the form $J_1 \times J_2 \times J_3$, where J_1, J_2 and J_3 are infinite subsets of $\mathbb{N} \cup \{0\}$.

2. $\mathbf{x} = (x_i)_{i \in I}$ is am indexed family.

The formal expression $\sum_{i \in I} x_i$ is called a formal infinite sum, or just an infinite sum.

When I is an infinite subset of $\mathbb{N} \cup \{0\}$, the formal sum $\sum_{i \in I} x_i$ is called a formal series. When $I = J_1 \times J_2$, where J_1 and J_2 are infinite subsets of $\mathbb{N} \cup \{0\}$, the formal sum $\sum_{i \in I} x_i$ is called a formal double series. When I is a set of the form $J_1 \times J_2 \times J_3$, where J_1, J_2 and J_3 are infinite subsets of $\mathbb{N} \cup \{0\}$, the formal sum $\sum_{i \in I} x_i$ is called a formal triple series.

And it should be clear what a "formal quadruple series", a "formal quintuple series", and so on, are.

1 Summing a formal sum

We now want to "sum" formal infinite sums $\sum_{i \in I} x_i$. That is, we want to assign a ***value*** to such sums.

The value of a sum $\sum_{i \in I} x_i$ is going to be defined in terms of the sums $\sum_{i \in F} x_i$ over finite subsets F of I. So it will be convenient to introduce a name for the set of all finite subsets of I. We define

$$\mathcal{F}(I) \stackrel{\text{def}}{=} \{F : F \subseteq I \wedge F \quad \text{is finite}\}.$$

So from now on we can say "$F \in \mathcal{F}(I)$" instead of "F is a finite subset of I".

In order to define the value of a sum $\sum_{i \in I} x_i$ we will need to know what the sums $\sum_{i \in F} x_i$ mean when $F \in \mathcal{F}(I)$. So we must confine ourselves to sums of families $(x_i)_{i \in I}$ such that *the finite sums* $\sum_{i \in F} x_i$ *make sense*. This means that we need to work with families $(x_i)_{i \in I}$ of objects x_i that belong to some set (or "number system") *where addition is possible*. There are many such systems (for example: the real numbers, the set of vectors in $\mathbb{R}^2$ and, more generally, the vectors in d-dimensional space $\mathbb{R}^d$ for any natural number d, the set of all continuous functions on an inteval). And we will start with the simplest such system.

So from on, until further notice, we will work with sums $\sum_{i \in I} x_i$ of families $(x_i)_{i \in I}$ *of real numbers*, that is, families $(x_i)_{i \in I}$ such that $x_i \in \mathbb{R}$ for every $i \in I$.

Our task is then

1. to assign a value to every formal sum $\sum_{i\in I} x_i$ of a family of real numbers, if such a thing is possible,

or

2. if we cannot assign a value to *every* formal sum $\sum_{i\in I} x_i$ of a family of real numbers, then at least assign a value to the formal sum $\sum_{i\in I} x_i$ for as many formal sums $\sum_{i\in I} x_i$ as possible.

It turns out that we can do this for many sums, but not for all of them.

But, before we go ahead and do that, we should understand what the problem is. If it was just a matter of "assigning a value to sums $\sum_{i\in I} x_i$", why can't we just assign, for example, the value zero to every sum? (This is a free country after all, and we have the right to name anything any way we want.)

You may answer that for *finite* sums we already know how to assign a value, and we should not change that. (For example, the value of the formal sum $3 + 6 + (-23) + 18$ is 4. It would be silly to change that and suddenly decree that $3 + 6 + (-23) + 18 = 0$.)

So let us try something a little bit less stupid: let us decree that the value of the sum $\sum_{i\in I} x_i$ is what we already know it is when the set I is finite, and the value is zero when the set I is infinite. Isn't this wonderful? We have assigned a value to *every* formal sum of a family of real numbers! So we can declare victory and go home.

Naturally, this solution is still very silly. I am sure you agree with that, don't you? But we should try to understand *why* it is silly. *PLEASE THINK ABOUT THIS QUESTION*. I would like you be able to give an intelligent answer to this question. (I will give you my own answer later, but please don't look at it right now. Try to answer this on your own.)

2 Assigning values to infinite sums

Now, let me show you how to assign a value to sums $\sum_{i\in I} x_i$ in a "reasonable" way.

I am not going to do it for *all* sums, but I will do it for *some* sums, and I claim that this is the best one can do. I have already shown to you how one could assign a value to *every* sum $\sum_{i\in I} x_i$, but the way I did it was very silly. So let me show you how we can solve the value-assignemnt problem in a "reasonable" way, and why we have to pay the price of allowing some sums $\sum_{i\in I} x_i$ *not* to have a value.

So, we are going to assign a value to lots of sums $\sum_{i\in I} x_i$. The value will be a real number, but will also be allowed[2] to be $+\infty$ or $-\infty$.

We want the value of a sum $\sum_{i\in I} x_i$ to be the "limit of the partial sums", whatever that means. A "partial sum" should be a sum $\sum_{i\in F} x_i$ over a *finite* subset of I. These sets should "get larger" in such a way that eventually every term x_i will "be included in the sum", that is, will occur in one of the finite sums $\sum_{i\in F} x_i$.

To do this rigorously, we give some precise definitions:

3 Exhausting sequences

Definition 1. Let I be a set. An exhausting sequence of subsets of I is a sequence $\mathbf{F} = (F_n)_{n=1}^{\infty}$ such that

1. Each F_n is a finite subset of I.
2. The sequence $\mathbf{F}$ is increasing[3], in the sense that
$$(\forall n \in \mathbb{N}) F_n \subseteq F_{n+1}\,.$$
3. For every $G \in \mathcal{F}(I)$, there exists $N \in \mathbb{N}$ such that
$$G \subseteq F_N\,.$$ □

Remark 1. If $G \in \mathcal{F}(I)$, then once we reach an N such that $G \subseteq F_N$, it will still be true for all n such that $n \geq N$ that $G \subseteq F_n$. (This is because $\mathbf{F}$ is

[2]If you don't like $+\infty$ or $-\infty$, then just say that, for you, the sum $\sum_{i\in I} x_i$ has a value if and only if it has a value r in my sense, and r is an ordinary real number.

[3]We have already talked in the course about increasing sequences of real numbers. (Remember that a sequence $(x_n)_{n=1}^{\infty}$ is increasing if $x_n \leq x_{n+1}$ for every $n \in \mathbb{N}$. This means that as n grows the numbers x_n become larger.) Here we are dealing with a different, but conceptually very similar, notion of "increasingness": this time, it is a sequence of *sets* that can be "increasing", meaning that, as n grows, the sets F_n get larger. One can talk in a similar way about sequences $(o_n)_{n=1}^{\infty}$ of objects of any kind being "increasing", as long as one has a way to *order* these objects, that is, a binary relation $\preceq$ which is transitive, in the sense that for any three objects a, b, c, $(a \preceq b \wedge b \preceq c) \Longrightarrow a \preceq c$. For exanple, suppose your binary relation is *reverse inclusion*, so "$A \preceq B$" means "$B \subseteq A$". Then an increasing sequence of sets with respect to this relation is a sequence $(A_n)_{n=1}^{\infty}$ of sets such that $A_2 \subseteq A_1$, $A_3 \subseteq A_2$, and so on.

increasing.) So every finite set is eventualy a subset of F_n, which means that *every finite subset G of I eventually gets included in the partial sums.* □

Remark 2. Why do we want the set I to be countable? The answer is as follows: *For a set I, there exists an exhausting sequence of finite sets of I if and only if I is fnite or countable.* Indeed, if an exhausting sequence $(F_n)_{n=1}^{\infty}$ exists, then I must be the union of all the F_n, and the union of a sequence of finite sets is finite or countably infinite, so I must be countable[4]. Conversely, if I is countable, then it is clear that I has an exhausting sequence $(F_n)_{n=1}^{\infty}$. (Proof: if I is finite, just take $F_n = I$ for every n. If I is countably infinite, let $f : \mathbb{N} \mapsto I$ be a bijection. Then define $F_n = \{f(k) : k \leq n\}$.)

Why is it important that there exist an exhausting sequence? Because if there is no exhausting sequence then all the statements of the form "for every exhausting sequence ..." will be true, vacuously. So for example it will be true that "for every exhausting sequence $\mathbf{F}$ the limit of the partial sums is r". And this will be true for *every* r, so every number will be the sum of the series $\sum_{i\in I} x_i$. Such a concept of "sum" would be totally useless. □

Definition 2. Let $S = \sum_{i\in I} x_i$ be a formal sums of real numbers, and let $\mathbf{F} = (F_n)_{n=1}^{\infty}$ be an exhausting sequence of subsets of I. The partial sums of S with respect to $\mathbf{F}$ are the real numbers $S_n^{\mathbf{F}}$ given by

$$S_n^{\mathbf{F}} = \sum_{i\in F_n} x_i \,. \qquad \square$$

We want to define the "value of the sum S" to be the limit of the partial sums $S_n^{\mathbf{F}}$ as $n \to \infty$. This, of course, poses a problem: *which exhausting sequence $\mathbf{F}$ shall we use?*

One possible answer would be to make some *ad hoc* choice of an exhausting sequence. This is what is done in the definition of "convergent series": when the set I is $\mathbb{N}$, we write our sum as $\sum_{n=1}^{\infty} x_n$ and define the sum of the series as the limit $\lim_{N\to\infty} \sum_{n=1}^{N} x_n$. In our terminology, this amounts to

[4]Let's be precise: in these notes, a finite set is a set S such that there exists a bijective map $f : \mathbb{N}_n \mapsto S$ for some $n \in \mathbb{N} \cup \{0\}$. (Here $\mathbb{N}_n \stackrel{\text{def}}{=} \{k \in \mathbb{N} : k \leq n\}$, so $\mathbb{N}_n$ is a set with n members.). A countably infinite set is a set S such that there exists a bijective map $f : \mathbb{N} \mapsto S$. A countable set is a set which is finite or countably infinite. Some authors prefer to call a set "countable" only if it is countably infinite in our sense, but other authors don't, and we have to make a choice, so we come down on the side of those who consider finite sets to be countable.

choosing the exhausting sequence $\mathbf{F} = (F_N)_{N=1}^{\infty}$, where $F_N = \mathbb{N}_N$, and then defining the sum of the series to be the limit of the partial sums arising from this sequence. A series that has a sum in this sense will be called convergent. This is useful, but has several disadvantages, that we will discuss later. (The most significant disadvantages are: (i) that in general it is not possible to rearrange the terms of a convergent series and get the same sum, and (ii) that convergent series cannot be multiplied.)

We are much more interested in the concept of *unconditionsl convergence*, which we will now proceed to define.

If I is a completely arbitrary set, there is no "natural" way to choose an exhausting sequence. So all the exhausting sequences are equally important, and we can avoid having to choose one such sequence by making sure that the limit of the partial sums is the same no matter which exhausting sequence we choose. This seems harder to achieve but, as we shall see, it is the way to go.

Definition 3. Let $S = \sum_{i\in I} x_i$ be a formal sum of real numbers. We say that the sum $\sum_{i\in I} x_i$ makes sense unconditionally if there exists an extended[5] real number r such that

(#) $\lim_{n\to\infty} \sum_{i\in F_n} x_i = r$ for every exhausting sequence $\mathbf{F} = (F_n)_{n=1}^{\infty}$. □

If the $\sum_{i\in I} x_i$ makes sense unconditionally, then the extended real number r is called the sum of the formal series $\sum_{i\in I} x_i$, and we write

$$\sum_{i\in I} x_i = r\,.$$

If in addition the extended real number r happens to be finite (that is, if r is a true real number, not $+\infty$ or $-\infty$), then we say that the formal series S converges unconditionally to r. □

Remark 3. You may not like this, but

> *When the sum $\sum_{i\in I} x_i$ makes sense unconditionally but it value is $+\infty$ or $-\infty$, we do **not** call the series "convergent". We call it "divergent".*

[5]Remember that an "extended real number" is an object that is either a real number, or $+\infty$, or $-\infty$. In other words, an extended real number is a member of $\overline{\mathbb{R}}$, the "extended real line". And $\overline{\mathbb{R}}$ is the set $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.

There are good reasons for this, and you will probably agree, after a while, that this is the right to do. In the meanwhile, if you are not convinced, just accept it "because the teacher says so". ☐

We now state and prove the two main results.

4 The sum of a family of nonnegative real numbers

Theorem 1. *Let $\mathbf{x} = (x_i)_{i\in I}$ be an indexed family of nonnegative real numbers. (That is, the x_i satisfy $x_i \in \mathbb{R}$ and $x_i \geq 0$ for every $i \in I$.) Assume that the index set I is countable. Then the sum $\mathbf{x} = (x_i)_{i\in I}$ makes sense unconditionally, and its value is either a nonnegative real number or $+\infty$.*

Proof. We want to prove that there exists an extended real number r such that Condition (#) of Definition 3 is satisfied.

The way to prove that an object of some kind exists is to exhibit one. And the way to do that is to pick one object and then prove that it satisfies the desired condition. (Naturally, *we have to figure out how to pick the right object*. If we just pick an "arbitrary object", that will not do. If you want to prove that[6] "a woman won the Fields Medal for Mathematics in 2014", picking an arbitrary woman will not do, because an arbitrary woman could be Taylor Swift, or Marie Curie, or Carly Fiorina, or Hillary Clinton, and none of these women has won the Fields Medal. What you have to do is figure out somehow that Maryam Mirzakhani won the medal in 2014, and she is a woman.)

So we choose r to be the supremum (i.e. the least upper bound[7]) of the set of all sums $\sum_{i\in G} x_i$, for all $G \in \mathcal{F}(I)$.

Now that we have defined an extended real number r in a perfectly correct way, we will prove that this number does the job we want it to do.

We have to prove that r satisfies (#). For this purpose, we let

$$\mathbf{F} = (F_n)_{n=1}^{\infty}$$

be an arbitrary exhausting sequence of finite subsets of I, and prove that

$$\lim_{n\to\infty} S_n^{\mathbf{F}} = r\,. \qquad (1.1)$$

[6]This is an existential statement: $(\exists x)$(x is a woman and x won the Fields Medal in 2014).

[7]Remember: in the extended real line $\overline{R}$, *every* subset X of $\mathbb{R}$ has a least upper bound. (And the least upper bound of X is an ordinary real number if X is nonempty and bounded above, and $+\infty$ if X is not bounded above.)

We begin by observing that the sequence $(S_n^{\mathbf{F}})_{n=1}^\infty$ is increasing, (Reason: for any $n \in \mathbb{N}$, $F_n \subseteq F_{n+1}$, which means that the sum $S_{n+1}^{\mathbf{F}}$ is equal to $S_n^{\mathbf{F}}$ plus some extra x_is. and since all the x_is are ≥ 0, we have $S_n^{\mathbf{F}} \leq S_{n+1}^{\mathbf{F}}$.)

Hence the limit $\lim_{n\to\infty} S_n^{\mathbf{F}}$ exists in $\overline{\mathbb{R}}$ (and is a nonnegative real number or $+\infty$). Let us call this limit $\tilde{r}$, so

$$\lim_{n\to\infty} S_n^{\mathbf{F}} = \tilde{r}\,. \tag{1.2}$$

(Notice the enormous difference between equations (1.1) and (1.2). Equation (1.1) is something we want to prove, and Equation (1.2) is something we know to be true.)

So what we have to do is prove that $\tilde{r} = r$. By definition, r is the least upper bound of the set of numbers $\sum_{i\in G} x_i$, for all $G \in \mathcal{F}(I)$. So in particular r is an upper bound for these numbers, meaning that

$$\sum_{i\in G} x_i \leq r \quad \text{for every} \quad G \in \mathcal{F}(I)\,.$$

Hence in particular

$$\sum_{i\in F_n} x_i \leq r \quad \text{for every} \quad n \in \mathbb{N}\,.$$

Since all the numbers in the sequence $(\sum_{i\in F_n} x_i)_{n=1}^\infty$ are $\leq r$, the limit $\tilde{r}$ of the sequence is also $\leq r$.

Hence

$$\tilde{r} \leq r\,.$$

Now we want to prove the opposite inequality, that is, $r \leq \tilde{r}$. To do this, we consider separately two possibilities: $r \in \mathbb{R}$ and $r = +\infty$. (The possibility that $r = -\infty$ does not arise because the x_i are nonnegative, so every finite sum $\sum_{i\in G} x_i$ is nonnegative, so the supremum of all these sums is nonnegative, so

$$r \geq 0\,.$$

Hence in particular r cannot be $-\infty$.)

1. The case when $r \in \mathbb{R}$. In this case, since r is the least upper bound of the numbers $\sum_{i\in G} x_i$, for $G \in \mathcal{F}(I)$, the number $r - \varepsilon$ is not an upper bound of these numbers, for any positive ε.

Let ε be an arbitrary positive number. Then $r-\varepsilon$ is not an upper bound for the numbers $\sum_{i\in G} x_i$, so we may pick a finite subset G of I such that

$$\sum_{i\in G} x_i > r-\varepsilon\,.$$

And now, the time has come to use the fact[8] that $\mathbf{F}$ is an exhausting sequence.

Since $\mathbf{F}$ is an exhausting sequence, for the the set G that we have picked there exists an $N\in\mathbb{N}$ such that $G\subseteq F_N$. Then the sum $\sum_{i\in F_N} x_i$ satisfies

$$\sum_{i\in F_N} x_i \geq \sum_{i\in G} x_i \geq r-\varepsilon\,.$$

But

$$\tilde{r} \geq \sum_{i\in F_N} x_i\,.$$

So

$$\tilde{r} \geq r-\varepsilon\,.$$

Since this is true for every positive ε, it follows that

$$\tilde{r} \geq r\,,$$

which is exactly what we wanted to prove.

So we have taken care of the case when r is a finite real number.

We now look at the other case.

2. The case when $r=+\infty$. In this case, since r is the least upper bound of the numbers $\sum_{i\in G} x_i$, for $G\in\mathcal{F}(I)$, no finite real number L can be an upper bound of these numbers.

Let L be an arbitrary real number. Then L is not an upper bound for the numbers $\sum_{i\in G} x_i$, so we may pick a finite subset G of I such that

$$\sum_{i\in G} x_i > L\,.$$

[8]We ***should*** use this somewhere, because if we didn't use it then the proof would have to be valid without this hypothesis, which means that the hypothesis isn't needed, which would meant that it was very stupid of me to put it in. YOU MUST REMEMBER THIS: ALWAYS MAKE SURE YOU USE ALL THE HYPOTHESES, AND THAT YOU TELL THE READER WHERE THE HYPOTHESES ARE USED.

Once again, we are going to use the fact that **F** is an exhausting sequence.

Since **F** is an exhausting sequence, for the the set G that we have picked there exists an $N \in \mathbb{N}$ such that $G \subseteq F_N$. Then the sum $\sum_{i \in F_N} x_i$ satisfies

$$\sum_{i \in F_N} x_i \geq \sum_{i \in G} x_i \geq L\,.$$

But

$$\tilde{r} \geq \sum_{i \in F_N} x_i\,.$$

So

$$\tilde{r} \geq L\,.$$

Since this is true for every $L \in \mathbb{R}$, it follows that

$$\tilde{r} = +\infty\,.$$

But we are assuming that $r = +\infty$. So

$$\tilde{r} = r\,,$$

which is exactly what we wanted to prove.

So we have proved that $\tilde{r} \geq r$, and we had proved before that $r \geq \tilde{r}$. Hence $\tilde{r} = r$. And this is true for an arbitrary exhausting sequence **F**. So Condition (#) of Definition 3 is satisfied, and our proof is finished. **Q.E.D.**

5 Absolutely convergent sums

We now want to prove our second main result. For this purpose, we need a definition:

Definition 4. A sum $\sum_{i \in I} x_i$ of a family of real numbers is said to be absolutely convergent if the sum of the absolute values of the x_i is finite, i.e., if

$$\sum_{i \in I} |x_i| < +\infty\,. \qquad \square$$

(Recall that the sum $\sum_{i \in I} |x_i|$ *always* makes sense, by Theorem 2, but could be finite or $+\infty$,)

Theorem 2. *Let* $\mathbf{x} = (x_i)_{i\in I}$ *be an indexed family of real numbers. Assume that the index set* I *is countable and that*

$$\sum_{i\in I} |x_i| < +\infty\,,$$

i.e., that the sum $\sum_{i\in I} x_i$ *is absolutely convergent.*

Then the sum $\mathbf{x} = (x_i)_{i\in I}$ *makes sense unconditionally and its value is a real number. (That is, the value of the sum is not* $+\infty$ *or* $-\infty$*.)*

Proof. We want to prove that there exists a real number r such that Condition (#) of Definition 3 is satisfied.

And, as in the previous proof, in order to prove that r exists we have to exhibit the number r and then prove that this number works.

If a is a real number, we define two numbers a_+ and a_- as follows:

$$a_+ = \begin{cases} a & \text{if} \quad a \geq 0 \\ 0 & \text{if} \quad a \leq 0 \end{cases}$$

$$a_- = \begin{cases} 0 & \text{if} \quad a \geq 0 \\ -a & \text{if} \quad a \leq 0 \end{cases}$$

The key properties of these numbers are the following:

(1.3) $$a_+ \geq 0 \text{ and } a_- \geq 0 \quad \text{for everry} \quad a \in \mathbb{R}\,,$$

(1.4) $$a = a_+ - a_- \quad \text{for everry} \quad a \in \mathbb{R}\,,$$

and

(1.5) $$|a| = a_+ + a_- \quad \text{for everry} \quad a \in \mathbb{R}\,,$$

(These facts are trivial consequences of the definition. You should have no difficulty proving them.)

Given this, we look at our formal sum $\sum_{i\in I} x_i$, and observe that

$$\sum_{i\in I} |x_i| < +\infty\,,$$

because we are assuming that the formal sum $\sum_{i\in I} x_i$ is absolutely convergent.

Now, the formal sum $\sum_{i\in I}(x_i)_+$ is a sum of nonnegative real numbers, because $(x_i)_+ \geq 0$ for every i. So the sum makes sense unconditionally, by Theorem 2. Let r_+ be the value of this sum, so $r_+ \in \overline{\mathbb{R}}$, and $0 \leq r_+ \leq +\infty$. It then turns out that $r_+ < +\infty$, because

1. $\sum_{i\in I}|x_i| < +\infty$ by the absolute convergence hypothesis,
2. $\sum_{i\in I}(x_i)_+ \le \sum_{i\in I}|x_i|$, because $(x_i)_+ \le |x_i|$ by (1.3)

So r_+ is a true (i.e. finite) real number, and

$$\sum_{i\in I}(x_i)_+ = r_+ \quad \text{unconditionally}\,.$$

That is, if $\mathbf{F} = (F_n)_{n=1}^{\infty}$ is any exhausting sequence of finite subsets of I, then

$$\lim_{n\to\infty}\sum_{i\in F_n}(x_i)_+ = r_+\,.$$

Exactly the same reasoning applies to the formal sum $\sum_{i\in I}(x_i)_-$. This sum is also a sum of nonnegative real numbers, because $(x_i)_- \ge 0$ for every i. So the sum makes sense unconditionally, by Theorem 2. Let r_- be the value of this sum, so $r_- \in \overline{\mathbb{R}}$, and $0 \le r_- \le +\infty$. It then turns out that $r_- < +\infty$, because

1. $\sum_{i\in I}|x_i| < +\infty$ by the absolute convergence hypothesis,
2. $\sum_{i\in I}(x_i)_- \le \sum_{i\in I}|x_i|$, because $(x_i)_- \le |x_i|$ by (1.3)

So r_- is a true (i.e. finite) real number, and

$$\sum_{i\in I}(x_i)_- = r_- \quad \text{unconditionally}\,.$$

That is, if $\mathbf{F} = (F_n)_{n=1}^{\infty}$ is any exhausting sequence of finite subsets of I, then

$$\lim_{n\to\infty}\sum_{i\in F_n}(x_i)_- = r_-\,.$$

Finally, we can exhibit the value that should be assigned to the sum $\sum_{i\in I} x_i$. We define

$$r = r_+ - r_-\,.$$

This is a real number, because both r_+ and r_- are real numbers.

Let us show that $\sum_{i\in I} x_i = r$ unconditionally. Let $\mathbf{F} = (F_n)_{n=1}^{\infty}$ be an arbitrary exhausting sequence of finite subsets of I. Then

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i\in F_n} x_i &= \lim_{n\to\infty}\sum_{i\in F_n}\Big((x_i)_+ - (x_i)_-\Big)\\
&= \lim_{n\to\infty}\Big(\sum_{i\in F_n}(x_i)_+ - \sum_{i\in F_n}(x_i)_-\Big)\\
&= \lim_{n\to\infty}\sum_{i\in F_n}(x_i)_+ - \lim_{n\to\infty}\sum_{i\in F_n}(x_i)_-\\
&= r_+ - r_-\\
&= r\,.
\end{aligned}$$

So we have shown that if $\mathbf{F} = (F_n)_{n=1}^{\infty}$ is an arbitrary exhausting sequence of finite subsets of I, then $\lim_{n\to\infty}\sum_{i\in F_n} x_i = r$. This proves that $\sum_{i\in I} x_i = r$ unconditionally, concluding our proof. **Q.E.D.**

6 The rearrangement theorem

Definition 5. A <u>permutation</u> of a set I is a function $\pi : I \mapsto I$ which is a bijection, that is, a map which is one-to-one and onto I. □

Definition 6. Consider a formal sum $S = \sum_{i\in I} x_i$ of a family of real numbers. A <u>rearrangement of the indices</u> of S is a permutation $\pi : I \mapsto I$ of the set I. A <u>rearrangement</u> of the sum S is a formal sum $\hat{S} = \sum_{i\in I} x_{\pi(i)}$, where π is a rearrangement of I. □

Theorem 3. *Let $S = \sum_{i\in I} x_i$ be a formal sum of a family of real numbers. Let π be a rearrangement of I.*

Assume that S is absolutely convergent. Then the rearranged sum $\hat{S}$ is also absolutely convergent, and the sums of S and $\hat{S}$ are equal.

Proof. Let S' be the sum $\sum_{i\in I} |x_i|$, and let $\hat{S}'$ be the sum $\sum_{i\in I} |x_{\pi(i)}|$.

Since both S' and $\hat{S}'$ are sums of nonnegative real numbers, we can compute the sums (which may be infinite) of S' and $\hat{S}'$ using any exhausting sequence we want.

So let us choose an exhausting sequence $\mathbf{F} = (F_k)_{k=1}^{\infty}$ and use it to compute the sum of S', and then make a clever choice of an exhausting sequence

$\mathbf{G} = (G_k)_{k=1}^{\infty}$ and use it to compute the sum of $\hat{S}'$. (The trick is to pick $\mathbf{G}$ is such a way that the partial sums of $\hat{S}'$ for $\mathbf{G}$ will be the same as the partial sums of S' for $\mathbf{F}$.)

We choose $\mathbf{G}$ by letting

$$G_k = \{i \in I : \pi(i) \in F_k\} \ \text{ for } \ k \in \mathbb{N}\,. \tag{1.6}$$

(That is, G_k is the set $\pi^{-1}(F_k)$.) Then it is easy to see that $\mathbf{G}$ is an exhausting sequence.

Then

$$\sum_{i\in I} |x_i| = \lim_{k\to\infty} \sum_{i\in F_k} |x_i| \tag{1.7}$$

and

$$\sum_{i\in I} |x_{\pi(i)}| = \lim_{k\to\infty} \sum_{i\in G_k} |x_{\pi(i)}|\,. \tag{1.8}$$

Furthemore, for each $i \in G_k$ the index $\pi(i)$ belongs to F_k, because the members on G_k are precisely those indices i for which $\pi(i) \in F_k$, according to (1.6). And each index j in F_k is equal to $\pi(i)$ for exactly one i. (The fact that i exists follows from the fact that π is onto, and the fact that i is unique follows because π is one-to-one.) And this i is in G_k, because of (1.6). Therefore, if we write the sum $\sum_{i\in F_k} |x_i|$ instead as $\sum_{j\in F_k} |x_j|$ (just changing the dummy variable i and writing j instead), and then write $j = \pi(i)$, so that when j takes all the values in F_k the index i takes all the values in G_k, we find

$$\begin{aligned}\sum_{i\in F_k} |x_i| &= \sum_{j\in F_k} |x_j| \\ &= \sum_{i\in G_k} |x_{\pi(i)}|\,.\end{aligned}$$

So

$$\sum_{i\in F_k} |x_i| = \sum_{i\in G_k} |x_{\pi(i)}| \quad \text{for every} \quad k \in \mathbb{N}\,.$$

Therefore

$$\lim_{k\to\infty} \sum_{i\in F_k} |x_i| = \lim_{k\to\infty} \sum_{i\in G_k} |x_{\pi(i)}|\,.$$

Hence, in view of (1.7) and (1.8), we have proved that

$$\sum_{i\in I} |x_i| = \sum_{i\in I} |x_{\pi(i)}|\,. \tag{1.9}$$

So the sum of S' is equal to the sum of $\hat{S}'$. Since we are assuming that S is absolutely convergent, the sum of S' is finite. Hence the sum of $\hat{S}'$is finite, so $\hat{S}$ is absolutely convergent.

We now want to prove that the sums of S and $\hat{S}$ are equal. This is done by repeating exactly the same calculation as before, this time without the absolute values.

We have

$$\sum_{i\in I} x_i = \lim_{k\to\infty} \sum_{i\in F_k} x_i \tag{1.10}$$

and

$$\sum_{i\in I} x_{\pi(i)} = \lim_{k\to\infty} \sum_{i\in G_k} x_{\pi(i)}\,. \tag{1.11}$$

Making the change of variable $j = \pi(i)$ as before, we find

$$\sum_{i\in F_k} x_i = \sum_{i\in G_k} x_{\pi(i)} \quad \text{for every} \quad k \in \mathbb{N}\,.$$

Therefore

$$\lim_{k\to\infty} \sum_{i\in F_k} x_i = \lim_{k\to\infty} \sum_{i\in G_k} x_{\pi(i)}\,.$$

Hence, in view of (1.10) and (1.11), we have proved that

$$\sum_{i\in I} x_i = \sum_{i\in I} x_{\pi(i)}\,. \tag{1.12}$$

So the sum of S is equal to the sum of $\hat{S}$. This concludes our proof. **Q.E.D.**

2 Series

We now take a look at the sums of ordinary series $\sum_{n\in\mathbf{N}} x_n$ of real numbers. (And we could equally well consider sums $\sum_{n\in J} x_n$.)

In this case, there is a "natural choice" of an exhausting sequence $\mathbf{F}$: we can take F_n to be the set $\{1, 2, \ldots, n\}$ (i.e., the set $\mathbb{N}_n$). (Or, if we are dealing with a sum $\sum_{n\in J} x_n$, we can take $F_n = \mathbb{N}_n \cap J$.)

We will define a series to be "convergent" if the partial sums corresponding to this particular sequence $\mathbf{F}$ converge.

Definition 7. A series $S = \sum_{n\in\mathbf{N}} x_n$ (that is, $S = \sum_{n=1}^{\infty} x_n$) converges to a real number r if

$$\lim_{N\to\infty} \sum_{n=1}^{N} x_n = r\,.$$

The series S converges, or is convergent, if there exists a real number r such that S converges to r. □

Remark 4. Definition 7 is exactly the definition of "convergence" of a series that was given in class when we began studying series, and is given in the book. □

How does this definition of convergence compare with the concept of "uncondditional convergence" introduced in Definition 3?

A partial answer is given by the following trivial observation:

Theorem 4.

1. *An unconditionally convergent series is convergent.*
2. *An absolutely convergent series*[9] *is unconditionally convergent, and in particular is convergent.*

Proof. This is completely trivial. An unconditionally convergent series is one for which the limit of the partial sums exists and is the same no matter which exhausting sequence is chosen. And a convergent series is one for which the limit of the partial sums using one particular exhausting sequence exists. So of course an unconditionally convergent series is convergent.

The other assertion is also evident. **Q.E.D.**

1 Conditionally convergent series

We now know that a convergent series S converges unconditionally if it is absolutely convergent. What happens if S converges but does not converge absolutely? Let us first give a name to this situation,

[9] A series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty} |x_n|$ is convergent. This definition is the special case for series of Definition 4. Since $\sum_{n=1}^{\infty} |x_n|$ is a series of nonegative terms, the sum makes sense uncondiitionally, so this series convergence in the sense of Definition 7 if and only if it converges unconditionally in the sense of Definition 3.

Definition 8. A conditionally convergent series is a series $\sum_{n=1}^{\infty} x_n$ that converges but does not converge absolutely. □

The following theorem describes exactly the very peculiar behavior of conditionally convergent series. It turns out that when a series is convergent but not absolutely convergent, we can reorder the terms of the series so as to make it converge to any real number we want, and even to $+\infty$ and $-\infty$.

Theorem 5. *Let $S = \sum_{n=1}^{\infty} x_n$ be a conditionally convergent series. Then, given any extended real number r, there is a rearrangement π of the index set $\mathbb{N}$ (i.e., a permutation $\pi : \mathbb{N} \mapsto \mathbb{N}$) such that the rearranged series $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges to r, in the sense that*

$$\lim_{N\to\infty} \sum_{n=1}^{N} x_{\pi(n)} = r\,.$$

The proof will be included in the next version of these notes.

2 Multiplication of series

Suppose

$$S = \sum_{n=0}^{\infty} x_n$$

and

$$T = \sum_{n=0}^{\infty} y_n$$

are two series of real numbers. We define the product of S and T to be the series

$$ST = \sum_{n=0}^{\infty} z_n\,,$$

where

$$z_n = \sum_{j=0}^{n} x_j y_{n-j}\,,$$

that is

$$z_n = x_0y_n + x_1y_{n-1} + x_2y_{n-2} + \cdots + x_{n-1}y_1 + x_ny_0\,.$$

Theorem 6. *If the series S and T are absolutely convergent, then the product series ST is absolutely convergent as well, and the sum of the product series ST is the product of the sums of the series S and T, that is,*

$$\sum_{n=0}^{\infty} z_n = \Big(\sum_{n=0}^{\infty} x_n\Big)\cdot\Big(\sum_{n=0}^{\infty} y_n\Big)\,. \tag{2.13}$$

Proof. Let us give names to the sums of the series S and T, by defining

$$s = \sum_{n=0}^{\infty} x_n\,,\quad \text{and}\quad t = \sum_{n=0}^{\infty} y_n\,.$$

Let U be the double series $\sum_{m,n} x_m y_n$, and let V be the double series $\sum_{m,n} |x_m y_n|$.

Since V is an infinite sum of nonnegative real numbers, we can compute the sum of V (which may be a nonnegative real number or $+\infty$) using any exhausting sequence we want. So we use the sequence

$$\mathbf{F} = (F_k)_{k=1}^{\infty} \tag{2.14}$$

given by

$$F_k = \{(m,n)\in(\mathbb{N}\cup\{0\})\times(\mathbb{N}\cup\{0\}) : m\le k \text{ and } n\le k\}\,. \tag{2.15}$$

We get

$$\begin{aligned}
\sum_{m,n}|x_m y_n| &= \sum_{m,n}|x_m|\cdot|y_n| \\
&= \lim_{k\to\infty}\sum_{(m,n)\in F_k}|x_m|\cdot|y_n| \\
&= \lim_{k\to\infty}\Big(\sum_{m=0}^{k}\sum_{n=0}^{k}|x_m|\cdot|y_n|\Big) \\
&= \lim_{k\to\infty}\Big(\Big(\sum_{m=0}^{k}|x_m|\Big)\cdot\Big(\sum_{n=0}^{k}|y_n|\Big)\Big) \\
&= \Big(\lim_{k\to\infty}\sum_{m=0}^{k}|x_m|\Big)\cdot\Big(\lim_{to\infty}\sum_{n=0}^{k}|y_n|\Big) \\
&= \Big(\sum_{m=0}^{\infty}|x_m|\Big)\cdot\Big(\sum_{n=0}^{\infty}|y_n|\Big)\,.
\end{aligned}$$

Since the series S and T are absolutely convergent, the sums $\sum_{m=0}^{\infty}|x_m|$ and $\sum_{n=0}^{\infty}|y_n|$ are finite. It then follows that

$$\sum_{m,n}|x_m y_n| < +\infty\,. \tag{2.16}$$

So the double series U is absolutely convergent. Hence the sum r of this double series is a real number, and this number can be computed using any exhausting sequence we like.

First let us compute r using the exhausting sequence $\mathbf{F}$ defined in (2.14), (2.15). We find

$$\begin{aligned}
r &= \sum_{m,n} x_m y_n \\
&= \sum_{m,n} x_m y_n \\
&= \lim_{k\to\infty} \sum_{(m,n)\in F_k} x_m y_n \\
&= \lim_{k\to\infty} \Big(\sum_{m=0}^{k}\sum_{n=0}^{k} x_m y_n\Big) \\
&= \lim_{k\to\infty} \Big(\Big(\sum_{m=0}^{k} x_m\Big)\cdot\Big(\sum_{n=0}^{k} y_n\Big)\Big) \\
&= \Big(\lim_{k\to\infty}\sum_{m=0}^{k} x_m\Big)\cdot\Big(\lim_{to\infty}\sum_{n=0}^{k} y_n\Big) \\
&= \Big(\sum_{m=0}^{\infty} x_m\Big)\cdot\Big(\sum_{n=0}^{\infty} y_n\Big) \\
&= st\,.
\end{aligned}$$

Next we compute r using the exhausting sequence

$$\mathbf{G} = (G_k)_{k=1}^{\infty}$$

given by

$$G_k = \{(m,n)\in(\mathbb{N}\cup\{0\})\times(\mathbb{N}\cup\{0\}) : m+n\le k\}\,.$$

This yields

$$
\begin{aligned}
r &= \sum_{m,n} x_m y_n && (2.17)\\
&= \lim_{k\to\infty} \sum_{(m,n)\in G_k} x_m y_n && (2.18)\\
&= \lim_{k\to\infty} \sum_{m+n\le k} x_m y_n && (2.19)\\
&= \lim_{k\to\infty} \sum_{j=0}^{k} \sum_{m+n=j} x_m y_n && (2.20)\\
&= \lim_{k\to\infty} \sum_{j=0}^{k} \Big(\sum_{m=0}^{j} x_m y_{j-n}\Big) && (2.21)\\
&= \lim_{k\to\infty} \sum_{j=0}^{k} z_j && (2.22)\\
&= \sum_{j=0}^{\infty} z_j\,. && (2.23)
\end{aligned}
$$

Therefore

$$r = \sum_{j=0}^{\infty} z_j\,.$$

So we have proved that the product series ST is convergent, and its sum is equal to st.

Hence we have proved everything we want, except for the absolute convergence of the product series ST. Let us now prove that.

Let

$$w_n = \sum_{m=0}^{n} |x_m|\cdot|y_{n-m}|\,.$$

It is then clear that

$$
\begin{aligned}
|z_n| &= |\sum_{m=0}^{n} x_m y_{n-m}|\\
&\le \sum_{m=0}^{n} |x_m|\cdot|y_{n-m}|\\
&= w_n\,.
\end{aligned}
$$

So $|z_n| \le w_n$, and then

$$\sum_{n=0}^{\infty} |z_n| \le \sum_{n=0}^{\infty} w_n \,. \tag{2.24}$$

To prove that the series ST is absolutely convergent, we have to prove that

$$\sum_{n=0}^{\infty} |z_n| < +\infty \,, \tag{2.25}$$

and this will follow from (2.25) if we prove that

$$\sum_{n=0}^{\infty} w_n < +\infty \,. \tag{2.26}$$

So all we need to do is prove (2.25).

In order to prove (2.26), we repeat the calculation we did in (2.17), (2.18), (2.19), (2.20), (2.21), (2.22), (2.23), "putting absolute values everywhere":

$$\begin{aligned}
\sum_{m,n} |x_m|\cdot|y_n| &= \lim_{k\to\infty} \sum_{(m,n)\in G_k} |x_m|\cdot|y_n| && (2.27)\\
&= \lim_{k\to\infty} \sum_{m+n\le k} |x_m|\cdot|y_n| && (2.28)\\
&= \lim_{k\to\infty} \sum_{j=0}^{k} \Big(\sum_{m=0}^{j} |x_m|\cdot|y_{j-n}|\Big) && (2.29)\\
&= \lim_{k\to\infty} \sum_{j=0}^{k} w_j && (2.30)\\
&= \sum_{j=0}^{\infty} w_j \,. && (2.31)
\end{aligned}$$

(Notice that (2.27), (2.28), (2.29), (2.30), (2.31) is exactly the same calculation as (2.17), (2.18), (2.19), (2.21), (2.22), (2.23), except that we have put absolute values everywhere.) It follows that the sum $\sum_{j=0}^{\infty} w_j$ is equal to the sum $\sum_{m,n} |x_m|\cdot|y_n|$, which we know is finite thanks to (2.16). So $\sum_{j=0}^{\infty} w_j$ is finite, and we have proved (2.26). And, since (2.26) was the only thing missing to complete our proof, we have, finally, completed the proof. **Q.E.D.**

3 An example: the exponential function

The exponential function is defined as follows: for each real number x, we let

$$e^x \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!}\,. \tag{2.32}$$

To make sure that this is well defined, we have to prove that for every x the series of (2.32) is convergent. We prove a stronger result, namely, that

Theorem 7. *For every real number x, the series of (2.32) is absolutely convergent.*

Proof. Fix x. Let N be a natural number such that $|x| < N$. Then

$$\begin{aligned} \frac{|x|^{N+1}}{(N+1)!} &= \frac{|x|^N}{N!} \times \frac{|x|}{N+1} \\ &= \frac{|x|^N}{N!}\rho\,, \end{aligned}$$

where

$$\rho = \frac{|x|}{N+1}\,.$$

It is clear that $0 \leq \rho < 1$. And then

$$\begin{aligned} \frac{|x|^{N+2}}{(N+2)!} &= \frac{|x|^{N+1}}{(N+1)!} \times \frac{|x|}{N+2} \\ &\leq \frac{|x|^{N+1}}{(N+1)!}\rho \\ &\leq \frac{|x|^N}{(N)!}\rho^2\,. \end{aligned}$$

Continuing in this way, we see that

$$\frac{|x|^{N+k}}{(N+k)!} = \frac{|x|^N}{(N)!} \times \rho^k \quad \text{for} \quad k \in \mathbb{N}\,.$$

It follows that

$$\sum_{n=N}^{\infty} \frac{|x|^n}{n!} \;\leq\; \sum_{n=N}^{\infty} C\rho^{n-N}$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} C\rho^n \\ &= \frac{C}{1-\rho} \\ &< +\infty\,, \end{aligned}$$

where

$$C = \frac{|x|^N}{N!}\,.$$

Since the sum $\sum_{n=0}^{N-1} \frac{|x|^n}{n!}$ is obviously finite, we conclude that the series $\sum_{n=0}^{\infty} \frac{|x|^n}{n!}$ is convergent, so the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent. **Q.E.D.**

So the exponential e^x is now well defined for every real number x.

Theorem 8. *If x and y are real numbers, then*

$$e^{x+y} = e^x e^y\,. \tag{2.33}$$

Proof. Let S be the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, and let T be the series $\sum_{n=0}^{\infty} \frac{y^n}{n!}$.

Since both series are absolutely convergent, Theorem 6 tells us that the product series ST is absolutely convergent, and the sum of ST is the product $e^x e^y$.

We now have to compute the product series. By definition,

$$ST = \sum_{n=0}^{\infty} z_n\,,$$

where

$$z_n = \sum_{j=0}^{n} \frac{x^j}{j!}\frac{y^{n-j}}{(n-j)!}\,,$$

so that

$$z_n = \frac{1}{n!}\sum_{j=0}^{n} \frac{n!}{j!(n-j)!}x^j y^{n-f}\,.$$

It then follows from the Binomial formula that

$$z_n = \frac{1}{n!}(x+y)^n\,.$$

Hence

$$e^x e^y = \sum_{n=0}^{\infty} \frac{1}{n!}(x+y)^n\,,$$

that is,

$$e^x e^y = e^{x+y}\,.$$

This is exactly the desired result. **Q.E.D.**

Having proved Theorem 8, one can prove easily some important properties of the exponential function:

Theorem 9.

1. *$e^0 = 1$,*
2. *$e^x > 0$ for every real number x,*
3. *$e^x > 1 + x$ if $x > 0$,*
4. *$e^x < \frac{1}{1+|x|}$ if $x < 0$.*
5. *the exponential function is strictly increasing, that is*

$$e^x < e^y \quad \text{whenever} \quad x, y \in \mathbb{R} \quad \text{and} \quad x < y\,.$$

Proof. The fact that $e^0 = 1$ follows by plugging in $x = 0$ in the formula for e^x.

If $x \in \mathbb{R}$, then

$$\begin{aligned} e^x &= e^{\frac{x}{2}+\frac{x}{2}} \\ &= e^{\frac{x}{2}} e^{\frac{x}{2}} \\ &= \left(e^{\frac{x}{2}}\right)^2, \end{aligned}$$

so e^x is the square of a real number, and then $e^x \geq 0$. But e^x cannot be $= 0$, because $e^x e^{-x} = e^0 = 1 \neq 0$. So $e^x > 0$.

If $x > 0$, then the series for e^x is a sum of positive numbers, so e^x is larger than the sum of the first two terms, i.e., $e^x > 1 + x$.

If $x < 0$, then $-x > 0$, so $e^{-x} > 1 + (-x) = 1 + |x|$. Since $e^x = \frac{1}{e^{-x}}$, we find $e^x < \frac{1}{1+|x|}$.

To prove that the function $x \mapsto e^x$ is increasing, pick x, y such that $x < y$. Then

$$e^y = e^{x+(y-x)} = e^x e^{y-x} \,.$$

Since $y - x > 0$, it follows that $e^{y-x} > 1 + (y - x) > 1$. So $e^x e^{y-x} > e^x$, that is, $e^y > e^x$.

This completes our proof. **Q.E.D.**

The stage is now set for defining the natural logarithm function ln. All thar is misssing is proving that the exponential function is continuous. If we could prove this, then it would follow that for every $y \in \mathbb{R}$ such that $y > 0$ there exists a unique $x \in \mathbb{R}$ such that $e^x = y$. (The existence of x follows by comsidering first the case when $y > 1$. In that case $e^0 = 1 < y$, and $e^y > 1 + y > y$, so by the Intermediate Value Theorem there is an x between 0 and y such that $e^x = y$. If $y < 1$ then $\frac{1}{y} > 1$, so we may find x such that $e^x = \frac{1}{y}$, and then $e^{-x} = y$. The proof of uniqueness is easy.)

So our next step in the study of the exponential function has to be proving that this function is continuous.