MATHEMATICS 300 — SPRING 2015

Introduction to Mathematical Reasoning

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INSTRUCTOR'S NOTES

LECTURE NO. 2

1 The real number system

We now start our systenatic study of number systems, by looking at the real numbers.

1.1 The basic concepts of real number theory

To study of the real numbers, we will want to analyze several concepts and properties associated with them, such as, for example:

1. the operations of addition, subtraction, multiplication and division of real numbers,
2. the number zero,
3. the number one,
4. the order relation ("$<$"),
5. the square of a real number and, more generally, powers of all kinds,
6. the absolute value of a real number,
7. integers and natural numbers.

The approach we will use is to start with some **basic concepts and properties**, and then **define** all the other ones. The basic concepts and properties will be represented by symbols, such as 0, 1, $+$, $\times$, $<$, and all other concepts of the theory will be defined in terms of them, and new symbols will be introduced to represent them. So, for example, "2" is not one of the basic concepts, so we will have to define "2". (This is easy: we will just define 2 to be $1+1$.) And "absolute value" is not one of the basic concepts either, so we will have to define what "absolute value" means.

Here is the list of the basic concepts of real number theory:

1. the numbers 0 (zero), and 1 (one),

2. the binary operations of addition, subtraction, multiplication and division:

   i. *addition* produces, for any two real numbers x, y, a real number $x+y$, called the *sum* of x and y,

   ii. *subtraction* produces, for any two real numbers x, y, a real number $x-y$, called the *difference* of x and y, (that is "x minus y"),

   iii. *multiplicartion* produces, for any two real numbers x, y, a real number xy, called the *product* of x and y; we also write[1] $x.y$, or $x \times y$, for the product xy.

   iv. *division* produces, for any two real numbers x, y such that $y \neq 0$, a number $x \div y$ (also written x/y, or $\frac{x}{y}$), called the *quotient* of x over y. Division is a *partially defined* operation, because the quotient $x \div y$ does not make sense for all possible real numbers x and y: $x \div y$ only makes sense when y is not equal to zero.

3. the *order relation* $<$ ("less than"): for any two real numbers x and y, x is either less than y or not. We write "$x < y$" to indicate that x is less than y.

Starting with these basic concepts, we will want to *define* all other concepts and properties of interest in the theory.

For example: what is the "absolute value" of a real number? Since "absolute value" is not one of our basic concepts, we have to *define* absolute value in terms of the basic concepts. And we cannot define the absolute value of a real number to be the "magnitude" of the number, because "magnitude" is not one of the basic concepts, so saying that "the absolute value

[1] This is especially useful when we are dealing with specific numbers represented by "numerals", i.e., symbols such as 23 or 3.72. If we want to write the product of 23 and 45, it is better not to write 2345, because this is the name of the number two-thousand three hundred and forty-five, rather than the product of 23 and 45. So it is much better to write 23×45.

of a real number is its magnitude" is meaningless, because we do not know what "magnitude" means[2].

Here is the correct way to define "absolute value".

DEFINITION. Given a real number x, the absolute value of x is the number $|x|$ defined as follows:

$$|x| = x \quad \text{if } x > 0\,, \tag{1.1.1}$$
$$|x| = -x \quad \text{if } 0 > x\,, \tag{1.1.2}$$
$$|x| = 0 \quad |\text{if } x = 0\,. \qquad \square \tag{1.1.3}$$

EXAMPLES:

1. $|5| = 5$, because $5 > 0$, so (1.1.1) applies.
2. $|-5| = 5$, because $0 > -5$, so (1.1.2) applies, and —$|-5| = -(-5) = 5$.
3. $|0| = 0$. □

1.2 The basic facts of real number theory. Part I: the field axioms

Now that we have described the basic symbols of real number theory, we list the basic properties of the concpets represented by those symbols. These basic properties are the *axioms* for the real numbers, that is, the facts that we take from granted and use as the starting point of the development of the theory. **Everything else has to be proved.**

We divide the list of axiuoms into two parts. First, in this subsection, we present the axioms about 0, 1, $+$, $\times$, $-$, and $/$. And then, in the next subsection, we list the axioms involving $<$.

The axioms about 0, 1, $+$, $\times$, $-$, and $/$ are called the field axioms, because any system of "numbers" in which special "numbers" 0 and 1, and operations $+$, $\times$, $-$, and $/$ that obey these axioms is called a field. (We will see later in the course examples of fields other than $\mathbb{R}$.)

And, before we actually list the axioms, we have to say a few words about equality ("$=$") and inequality ("$\neq$"), because these concepts will appear in the axioms. So we digress a little bit and talk about equality.

[2]Based on my own experience of teaching Math 300 many times, I can predict that, when asked to define "absolute value" in one of the midterms or the final exam, many students are going to write "the absolute value of a real number is its magnitude". **Please do not do that!**

1.2.1 Equality and inequality

If x and y are any objects (numbers, sets, people, whatever), we write "$x = y$" (and read this as "x is equal to y", or "x and y are equal") to indicate that x and y are the same object. And we write "$x \neq y$" to indicate that x and y are not the same, so "$x \neq y$" means exactly the same thing as "$\sim x = y$".

Equality obeys the following laws (called "equality axioms"):

EA1. (reflexive law of equality) If x is any object, then $x = x$.

EA2. (symmetry law of equality) If x, y are any objects, and $x = y$, then $y = x$.

EA3. (transitivity law of equality) If x, y, z are any objects, $x = y$, and $y = z$, then $x = z$.

In addition, equality satisfies the following "substitution of equals for equals" rule, that can be used in proofs:

RULE SEE: If

a. S is a statement containing, once or several times, a term[3] T,

b. U is another term,

c. we have $U = T$ or $T = U$ in an earlier step of our proof,

dc. we have S in an earlier step of our proof,

then we can assert, in a new step of our proof, a sentence obtained from S by substituting for T the term U, in asome or all the occurrences of T in S.

For example: if we have $2 = 1 + 1$ and $2 + 1 = 3$ in earlier steps, we can write $(1+1)+1 = 3$. And, if we have $2 = 1+1$ and $2+2 = 4$ in earlier steps, we can write $(1 + 11) + 2 = 4$, or $2 + (1 + 1) = 4$, or $(1 + 1) + (1 + 1) = 4$. (That is, we can substitute "$1 + 1$" for "2" in the first of the two 2s that occur in $2 + 2 = 4$, or in the second one, or in both.)

[3] "Terms" and "sentences" will be discussed in detail later. At this point, all you need to know is that a <u>term</u> is an expression that is the name of an object, and a <u>sentence</u> is an expression that makes an assertion that can be true or false. For example, "1", "$1 + 1$", "$2 + 3$", "$(7.43 + 22.04) \times 96$", "Mt. Everest", "Lady Gaga", and "The man who came to dinner yesterday evening" are terms, and "$2 + 2 = 4$", "Mt. Everest is taller than Mt. McKinley", "Lady Gaga sang together with Tony Bennett", and "The man who came to dinner yesterday evening didn't stay very long" are sentences.

And now we are ready to go back to the discussion of the real number axioms.

1.2.2 The list of the field axioms

FA1. (closure laws) If x, y are real numbers, then

- FA1.a. $x + y$, $x - y$, and xy are real numbers,
- FA1.2. if $y \neq 0$ then x/y is a real number.

FA2. (associative law of addition) If x, y, z are real numbers, then $(x + y) + z = x + (y + z)$.

FA3. (commutative law of addition) If x, y are real numbers, then $x + y = y + x$.

FA4. (associatve law of multiplication) If x, y, z are real numbers, then $(xy)z = x(yz)$.

FA5. (commutative law of multiplication) If x, y are real numbers, then $xy = yx$.

FA6. (distributive law of multiplication with respect to addition) If x, y, z are real numbers, then $x(y + z) = xy + xz$.

FA7. (subtraction axiom) If x, y are real numbers, then $(x - y) + y = x$.

FA8. (division axiom) If x, y are real numbers, and $y \neq 0$, then $(x \div y).y = x$.

FA9. (additive identity law) If x is a real number, then $x + 0 = x$.

FA10. (multiplicative identity law) If x is a real number, then $x \times 1 = x$.

FA11. $0 \neq 1$[4]

[4] You may think that this is "obvious". But the point is this: if it is indeed obvious, then you should be able to prove that it is true, and it turns out that *you cannot prove that* $0 \neq 1$ *from the other axioms*, so if you want it to be true you have to put it as an axiom. Also, remember that **nobody said that the axioms have to be sophisticated, nontrivial statements**. What the axioms have to be is **clearly, unquestionably true**. And "$0 \neq 1$" is clearly true, so that's a good reason for putting it as an axiom.

1.3 A brief detour into Logic: quantifiers

If you look at any of the 11 axioms listed in the previous subsection, you will see that their statement is made in a mixture of ordinary language and formulas. For example, Axiom FA6 says: "If x, y, z are real numbers, then $x(y + z) = xy + xz$". This uses the formula $x(y + z) = xy + xz$, and also English words.

It turns out that mathematicians, and logicians, have invented "formal languages", in which you can say *everything* with formulas, without using any words. It is going to be very important for us to learn formal language, and to be able to translate from English to formal language and back.

REMARK. Why is this important? There are several reasons, and we will talk about them later. For the moment, let me give you just one reason. ***In formal language, you are obliged to be absolutely precise.*** If you are saying something in English, and you cannot translate it into formal language, it means that you really do not know what it is exactly that you are trying to say. For example, the statements "8 is a small number", or "x is a number", cannot be translated into formal language, and this is an indication that you have to think some more, figure out exactly what you are trying to say, and once you know precisely what it is that you want to say, then you will be able to say it in formal language.

Why is "8 is a small number" not precise? The answer is, simply, that in Mathematics there is no such things as a "small number". Smallness depends very much on the context. If you are talking about the number of people who attended a concert, then 8 is a small number. But if you are talking about the number of people who claim to have won a disputed election, then 8 is not at all a small number.

Similarly, "x is a number" is not a precise statement. (See Lecture 1, Page 9.) You can say "x is a real number" in formal language, by saying "$x \in \mathbb{R}$". Or you can say "x is an integer", by saying "$x \in \mathbb{Z}$". So one way for you to realize that you are not supposed to say "x is a number", is to try to say it in formal language and see that you cannot do it. □

We now begin our discussion of the symbols of formal language by talking about the two quantifiers: existential and universal. (We already talked about existential quantifiers in Lecture 1, pages 10-11.)

The symbols $\exists$ and $\forall$ are the ***quantifier symbols***: " $\exists$ " is the ***existential quantifier symbol***, and " $\forall$ " is the ***universal quantifier symbol***.

Using these symbols, we can form ***quantifiers***. An ***existential quantifier*** is an expression "$(\exists x)$" or "$(\exists x \in S)$" (if S is a set). "$(\exists x)$" is an ***unrestricted existential quantifier***, and ''$(\exists x \in S)$" is a ***restricted existential quantifier***.

Similarly, a ***universal quantifier*** is an expression "$(\forall x)$" or "$(\forall x \in S)$" (if S is a set). "$(\forall x)$" is an ***unrestricted universal quantifier***, and ''$(\forall x \in S)$" is a ***restricted universal quantifier***.

Quantifiers are read as follows:

1. "$(\exists x)$" is read as

   – "there exists x such that"

   or

   – "for some x"

   or

   – "it is possible to pick x such that".

2. "$(\exists x \in S)$" is read as

   – "there exists x belonging to S such that"

   or

   – "there exists a member x of S such that"

   or

   – "for some x in S"

   or

   – "it is possible to pick x in S such that"

   or

   – "it is possible to pick a member x of S such that"

3. "$(\forall x)$" is read as

   – "for all x"

or

- “for every x ”

or

- “given any x”

or

- “no matter who x is”

4. “$(\forall x \in S)$” is read as

- “for all x in S”

or

- “for every x in S ”

or

- “given any x in S”

or

- “no matter who x in S is”

or

- “for all members x of S”

or

- “for every member x of S ”

or

- “given any member x of S”

or

- “for all x belonging to S”

or

- "for every x belonging to S "

or

- "given any x belonging to S".

1.4 The connectives "$\wedge$" (meaning "and"), "$\vee$" (meaning "or") , and "$\Longrightarrow$" (meaning "implies")

The symbol "$\wedge$" is the ***conjunction symbol***, and means "and". So, for example, if P is the sentence "today is Friday" and Q is the sentence "tomorrow is Saturday", then "$P \wedge Q$" stands for the sentence "today is Friday and tomorrow is Saturday".
The symbol "$\vee$" is the ***disjunction symbol***, and means "or". So, for example, if P is the sentence "today is Friday" and Q is the sentence "today is Saturday", then "$P \vee Q$" stands for the sentence "today is Friday or today is Saturday".
The symbol "$\Longrightarrow$" is the ***implication symbol***, and means "implies". A sentence "$P \Longrightarrow Q$" is read as "P implies Q", or "If P then Q". So, for example, if P is the sentence "today is Friday" and Q is the sentence "tomorrow is Saturday", then "$P \Longrightarrow Q$" stands for the sentence "If today is Friday then tomorrow is Saturday".

REMARK. Notice that "$\wedge$" and "$\Longrightarrow$" are very different. For example, the sentence "today is Friday and tomorrow is Saturday" is true only if today is Friday. On the other hand, the sentence "If today is Friday then tomorrow is Saturday" is true no matter what day it is today. (Think of "If today is Friday then tomorrow is Saturday" as meaning "If today was Friday then tomorrow would be Saturday". This is always true, even is today happens to be Tuesday. If you are not convinced, wait. Implication will ve discussed later.)

1.4.1 The field axioms restated in formal language

Using universal quentifier, conjuctions and implications, let us restate all the field axioms into formal language. Here they are:

The field axioms for $\mathbb{R}$.

FA1. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(x + y \in \mathbb{R} \wedge x - y \in \mathbb{R} \wedge xy \in \mathbb{R} \wedge (y \neq 0 \Longrightarrow x \div y \in \mathbb{R}))$.

FA2. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})(x + y) + z = x + (y + z)$.

FA3. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})x + y = y + x$.

FA4. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})(xy)z = x(yz)$.

FA5. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})xy = yx$.

FA6. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})x(y + z) = xy + xz$.

FA7. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(x - y) + y = x$.

FA8. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(y \neq 0 \Longrightarrow (x \div y).y = x)$.

FA9. $(\forall x \in \mathbb{R})x + 0 = x$.

FA10. $(\forall x \in \mathbb{R})x.1 = x$.

FA11. $0 \neq 1$.

1.5 How to prove universal sentences, ccnjunctions, disjunctions, and implications

We now start presenting the rules of Logic, that govern proofs. Eventually, when we have gone through the full list, it will turn out that there are exactly 15 rules, all of which are very easy to remember and understand.

We have already seen one rule (Rule SEE). We now present three more.

The rule for proving a universal sentence (Rule $\forall_{prove}$)

If $P(x)$ is a sentence involving the variable x, then:

1. If, starting with "Let x be arbitrary" you prove $P(x)$, then you can conclude that $(\forall x)P(x)$.
2. If S is a set, and starting with "Let x be an arbitrary member of S" you prove $P(x)$, then you can conclude that $(\forall x \in S)P(x)$.

The meaning of "arbitrary" is explained in the box in the next page.

The rule for proving a conjunction (Rule $\wedge_{prove}$)

If P, Q are sentences, and you prove P and you prove Q. then you csn go to $P \wedge Q$.

REMARK. This is the stupidest rule in the world! You may wonder "what is the point of such a rule?" But you cannot dispute that it is reasonable rule! Of course, if you know that "today is Friday" and you also know that "tomorrow is Saturday", then you have no doubt that "today is Friday and tomorrow is Saturday" is true. So you should have no problem accepting (and remembering) this rule. You may not understand wjhy it is needed. So let me tell you why. Suppose it was a computer doing proofs, rather than a human being like you. Suppose the computer is told that today is Friday and then it is told that tomorrow is Saturday. How will the computer know that it can write "today is Friday and tomorrow is Saturday". It won't, unless you tell it. Computers do not "know" anything. If you want the computer to "know" that once it knows that "today is Friday" and also that "tomorrow is Saturday", then it can write "today is Friday and tomorrow is Saturday", then you have to tell the computer. In other words, you have to input Rule $\wedge_{prove}$ into the computer. Proofs are mechanical manipulations of strings of sysmbols, so they should be doable by a computer. So Rule $\wedge_{prove}$ is needed.

Arbitrary objects

In order to prove that a property P is true for every object in some set S, we pick an "arbitrary" member of S, call it x or "Billy"), and prove that P holds for x. (You could call it any name you want: y, or a, or α, or "BIlly". The name doesn't matter, except for one thing: you *cannot* use as a name a symbol that is already the name of something else. If you manage to do this, then you may conclude that P is true for every member of S, This is called the *universal generalization rule*, and will be widely used throughout this course, because it is one of the most important logical rules.

An "arbitrary" member of S a member of S that we can work with and reason about, but we don't know which specific object it is, and for all we know could be any member of S. You can think of this as follows: an "arbitrary" member of S is a memmber of S that has been given to you by an imaginary character called the CAT ("creator of arbitrary things"), who brings this object over to you inside a sealed envelope, so you have the object in your hands and can reason about it, but you don't know which member of S it is, and could turn out to be any member of S. Therefore, whatever you say about this object had better be true of *every* member of S, because if there is just one member of S for which what you say isn't true, then the "arbitrary" object could turn out to be that object.

Another way to think about "arbitrary" objects is this: imagine that x is a member of S that is going to be brought to you by the CAT *later*, after you have written your proof. So when you write your proof whatever you say about x had better be true for *all* members of S, because if there is one member of S for which what you say isn't true (such a member of S is called a "counterexample") then that member of S could be precisely the one that the CAT gives you.

You can even go farther, and think that the CAT is very mean, and wants to prove you wrong. So the CAT will look for a counterexample and will give you that counterexanple. The only way you can outsmart the CAT is by makibg sure that what you say is true *for all members of* S, so that the CAT cannot find a counterexample.

The rule for proving a disjunction (Rule $\vee_{prove}$)

Suppose P and Q are sentences, and you want to prove $P \vee Q$. Here is what you can do. You look at the two possible case, when P is true and when P is false. If P is true then of course $P \vee Q$ is true, so we are O.K. So all we have to do is look at the other case, when P is false, and prove that in that case Q is true.
So here is the rule: if assuling that P is false you cna prove Q, gthen you can go to $P \vee Q$.

The rule for proving an implication (Rule $\Longrightarrow_{prove}$)

Suppose P, Q are sentences. Suppose you can start a proof with "Assume P", and you prove Q. Then you csn go to $P \Longrightarrow Q$.

EXAMPLE. Say you are a Martian who just landed on Earth, you know nothing about the days of the week, and you want to prove that "If today is Friday then tomorrow is Saturday". To apply Rule $\Longrightarrow_{prove}$, you would begin by "assuming that today is Friday." This means that you would imagine that today is Friday, and see what would happen in that case. For example, you could go to a public library and look at lots of newspapers that have a Friday date, and you would see that every time such a paper talks about the following day it says something like "tomorrow is Saturday." Then you would be reasonably confident that the sentence "If today is Friday then tomorrow is Saturday" is true. And it would not matter whether today is Friday or not.

1.6 Some simple definitions and proofs using the field axioms

Warning. You are going to find the next few profs extremely silly. For example, we are going to prove that 2+2 = 4, and you will probably complain, saying: "That's silly! I already know that, so what's the point of proving it?" I have three answers to that. First answer: If it's not in the axioms, we have to prove it. Second anser: Think of this course as similar to a language course, in which you are leanring a new language. In, for example, a course of English as a foreign language, you would not start the course with Shakespeare. You would start with simple sentences like "the cat is on the mat", "my mom loves me", or "Jack and Jill went up the hill." Then, step by step, you would move on to harded, more complicated sentences, andmaybe by the end of the semester you would be reading Hamlet's soliloquy. In this course, we are doing the same thing. Prove that 2+2=4 is the equivalent of learning to read and write the stetement "the cat is on the mat." And, believe me, by the end of the semester we will be doing reall interesting proofs. □

In the study of real numbers, all the things we want to talk about that are not primitive concepts have to be introduced into the theory by defining them, i.e., by explaining what they mean.

One concept that does not appear in the list of basic concepts is the *negative* of a real number. (The "$-$" synbol does appear, but only in the context of the subtraction operation, which takes two numbers x, y and produces the number $x = y$. The "negative of" concept, as when we talk about the number $-x$ for a given number x, is diffeerent.)

DEFINITION 1. If x is a real number, then the <u>negative</u> of x is the number $-x$ given by

$$-x = 0 - x\,. \qquad \square$$

Notice that this defines $-x$ in terms of the basic concepts, because it involves "0" and "$-$" (the sense of difference of two numbers, not that of the negative of a number).

And we can do the same for the "inverse" of a nonzero real number.

DEFINITION 2. If x is a real number, and $x \neq 0$, then the <u>inverse</u> (or <u>multiplicative inverse</u>) of x is the number x^{-1} given by

$$x^{-1} = 1 \div x\,. \qquad \square$$

Again, this defines x^{-1} in terms of the basic concepts, because it involves "1" and division.

And now let us prove a few things.

THEOREM 1. (The cancellation law for addition.) If x, y, z are real numbers, and $x + y = x + z$, then $y = z$. (In formal language: $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})((x + y = x + z) \Longrightarrow y = z)$.)

PROOF. Let x, y, z be arbitray real numbers.

Assume that $x+y=x+z$. We want to prove that $y=z$.

We have

$$(-x)+(x+y)=(-x)+(x+y)\,, \tag{1.6.4}$$

because of Axiom EA1 (every object is equal to itself), applied to the object $(-x)+(x+y)$.

Since $x+y=x+z$, we can substitute $x+z$ for the second of the two $x+y$'s in (1.6.4), and conclude that

$$(-x)+(x+y)=(-x)+(x+z)\,. \tag{1.6.5}$$

The associative law of addition implies that

$$(-x)+(x+y)=((-x)+x)+y\,,$$

so we may susbtitute $((-x)+x)+y$ for $(-x)+(x+y)$ in (1.6.5), and conclude that

$$((-x)+x)+y=(-x)+(x+z)\,. \tag{1.6.6}$$

Similarly,

$$(-x)+(x+z)=((-x)+x)+z\,,$$

so

$$((-x)+x)+y=((-x)+x)+z\,. \tag{1.6.7}$$

Now, according to the definition of "negative", we have $-x=0-x$. So we may substitute $0-x$ for $-x$ in (1.6.7), and get

$$((0-x)+x)+y=((0-x)+x)+z\,. \tag{1.6.8}$$

Mext, according to Axiom FA7 (applied with x in the role of y, and 0 in the role of x), we have

$$(0-x)+x=0\,.$$

So we may substitute 0 for $(0-x)+x$ in (1.6.8), and get

$$0+y=0+z\,. \tag{1.6.9}$$

Finally, according to Axiom FA9[5] $0+y=y$ and $0+z=z$. So we may substitute y for $y+0$ and z for $z+0$ in (1.6.9), getting

$$y=z\,. \tag{1.6.10}$$

Q.E.D.

[5]To be precise, Axiom FA9 tells us that $y+0=y$ and $z+0=z$. To get the conclusions that $0+y=y$ and $0+z=z$. we need a couple of extra steps, using Axiom FA3 to conclude that $y+0=0+y$ and $z+0=0+z$, so , $0+y=y$ and $0+z=z$. **What we have done in this proof is something that we will keep doing from on on: skip steps that are trivial and obvious.**

THE SAME PROOF, WRITTEN MORE CONCISELY, SKIPPING LOTS OF TRIVIAL STEPS. Assume that $x + y = x + z$. We want to prove that $y = z$.

We have (thanks to Axiom EA1) object $(-x) + (x + y)$.

$$(-x) + (x + y) = (-x) + (x + y)\,. \tag{1.6.11}$$

Since $x + y = x + z$, we get

$$(-x) + (x + y) = (-x) + (x + z)\,. \tag{1.6.12}$$

Using the associative law of addition, we find

$$((-x) + x) + y = ((-x) + x) + z\,. \tag{1.6.13}$$

Since $-x = 0 - x$, (1.6.13) implies

$$((0 - x) + x) + y = ((0 - x) + x) + z\,. \tag{1.6.14}$$

According to Axiom FA7, $(0 - x) + x = 0$. So

$$0 + y = 0 + z\,. \tag{1.6.15}$$

Since $0 + y = y$ and $0 + z = z$, we get

$$y = z\,. \tag{1.6.16}$$

Q.E.D.

THEOREM 2. (The cancellation law for multiplication.) If x, y, z are real numbers, $x \neq 0$, and $xy = xz$, then $y = z$.

PROOF. YOU DO THIS ONE. (It's almost exactly the same as the previous proof.)

THEOREM 3. If $x \in \mathbb{R}$, then $x.0 = 0$. (In formal language, $(\forall x \in \mathbb{R})x.0 = 0$.)

REMARK. This proof is short and easy, but it involves a trick. So **you have to know the trick**, because if you are asked to write this proof in an exam[6] and you don't know the trick, you may not be able to figure it out on your own . This means that **you have to study this proof[7].**

[6] Which may very well happen, believe me! I *do* know what I am talking about!

[7] Actually, you should study all the proofs, but this particular one is tricky.

PROOF. Let x be an arbitrary real number. We apply Axiom EA1 to write

$$x.0 = x.0\,. \tag{1.6.17}$$

Then we use Axiom FA1 (with 0 in the role of x, to conclude that

$$0 + 0 = 0\,. \tag{1.6.18}$$

Then we use Rule SEE to substitute $0 + 0$ for 0 in one of the two sides of (1.6.17), getting

$$x.(0 + 0) = x.0\,. \tag{1.6.19}$$

Next we use the distributive law (Axiom FA6) to conclude that

$$x.(0 + 0) = x.0 + x.0\,. \tag{1.6.20}$$

Then, using Rule SEE again, we find

$$x.0 + x.0 = x.0\,. \tag{1.6.21}$$

But

$$x.0 + 0 = x.0\,. \tag{1.6.22}$$

Hence we obtain

$$x.0 + x.0 = x.0 + 0\,. \tag{1.6.23}$$

Now we use the cancellation law of addition (Theorem 1), with $x.0$ in the role of x, $x.0$ in the role of y, and 0 in the role of z, to conclude that

$$x.0 = 0\,. \tag{1.6.24}$$

Q.E.D.

THEOREM 4. If $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $xy = 0$, then $x = 0$ or $y = 0$.

PROOF. Either $x = 0$ or $x \neq 0$.

We are going to apply Rule $\vee_{prove}$. We wanrt to prove that $x = 0 \vee y = 0$. So we assume that $x \neq 0$ and set out to prove that $y = 0$. This will then tell us that $x = 0 \vee y = 0$.

Axiom FA8 (applied with 1 in the role of x, and x in the role of y, tells us that $(1 \div x).x = 1$. Also, Axiom FA10 tells us that $y = y.1$, and Axiom FA5 implies that $y.1 = 1.y$, so $y = 1.y$. Hence

$$\begin{aligned} y &= 1.y \\ &= (1 \div x).x).y \\ &= (1 \div x).(xy) \\ &= (1 \div x).0\,. \end{aligned}$$

But Theorem 3 says that any real number times zero is equal to zero, so $(1 \div x).0 = 0$. Since $y = (1 \div x).0$, it follows that $y = 0$.

Q.E.D.

Now we would like to talk about the number 2, 3, etc. So we give a few definitions:

DEFINITION 3. $2 = 1 + 1$. □

DEFINITION 4. $3 = 2 + 1$. □

DEFINITION 5. $4 = 3 + 1$. □

THEOREM 5. $2 + 2 = 4$.

PROOF. It follows from Axiom EA1 that

$$2 + 2 = 2 + 2\,. \tag{1.6.25}$$

Definition 3 tells us that $2 = 1 + 1$.

So (using Rule SEE) we may substitute $1 + 1$ for the last of the four 2's of Equation (1.6.25), and get

$$2 + 2 = 2 + (1 + 1)\,. \tag{1.6.26}$$

By the associative law of addition (Axiom FA2), $2 + (1 + 1) = (2 + 1) + 1$. So (using Rule SEE)

$$2 + 2 = (2 + 1) + 1\,. \tag{1.6.27}$$

But $2 + 1 = 3$, by Definition 4. Hence

$$2 + 2 = 3 + 1\,. \tag{1.6.28}$$

And $3 + 1 = 4$, by Definition 5. Therefore

$$2 + 2 = 4\,. \tag{1.6.29}$$

Q.E.D.

THEOREM 6. $2 \times 2 = 4$.

PROOF. It follows from Axiom EA1 that

$$2 \times 2 = 2 \times 2\,. \tag{1.6.30}$$

Definition 3 tells us that $2 = 1 + 1$. So (using Rule SEE) we may substitute $1 + 1$ for the last of the four 2's of Equation (1.6.30), and get

$$2 \times 2 = 2 \times (1 + 1)\,. \tag{1.6.31}$$

By the distributive law (Axiom FA6)), $2 \times (1 + 1) = 2 \times 1 + 2 \times 1$. So (using Rule SEE)

$$2 \times 2 = 2 \times 1 + 2 \times 1\,. \tag{1.6.32}$$

By Axiom FA10, $2 \times 1 = 2$. Hence

$$2 \times 2 = 2 + 2\,. \tag{1.6.33}$$

Finally, Theorem 5 tells us that $2 + 2 = 4$. So, using Rule SEE, we find

$$2 \times 2 = 4\,. \tag{1.6.34}$$

Q.E.D.

DEFINITION 6. $5 = 4 + 1$. □

DEFINITION 7. $6 = 5 + 1$. □

DEFINITION 8. $6 = 5 + 1$. □

THEOREM 7. $3 + 3 = 6$.

PROOF. YOU DO THIS ONE.

THEOREM 8. $3 \times 2 = 6$.

PROOF. YOU DO THIS ONE.