MATHEMATICS 300 — SPRING 2015

Introduction to Mathematical Reasoning

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INSTRUCTOR'S NOTES

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1 More on the natural numbers and induction

Recall that a natural number n is <u>even</u> if there exists $k \in \mathbb{N}$ such that $n = 2k$. Similarly n is <u>odd</u> if there exists $k \in \mathbb{N}$ such that $n = 2k - 1$.

Theorem 1. *Every natural number is either even or odd.*

Proof. Let $P(n)$ be the statement "n is either even or odd", for natural numbers n. We want to prove that $(\forall n \in \mathbb{N})P(n)$.

We will prove this by induction. For this purpose, we will prove that $P(1)$ and that $(\forall n)(P(n) \Longrightarrow P(n+1))$.

Base case. $P(1)$ is true, because $P(1)$ says that "1 is even or odd", and that is true because 1 is odd.

Inductive step. We prove that $(\forall n \in \mathbb{N})(P(n) \Longrightarrow P(n+1))$.

Let n ne an arbitrary natural number.

We want to prove that $P(n) \Longrightarrow P(n+1)$. To prove this, we assume $P(n)$ and prove $P(n+1)$.

So assume that $P(n)$ is true. Then n is even or n is odd. To prove that $P(n+1)$ is true, we consider the cases when n is even and when n is odd. If n is even, then we may pick $k \in \mathbb{N}$ such that $n = 2k$. Then $n+1 = 2k+1 = 2(k+1)-1$, and $k+1 \in \mathbb{N}$. So $n+1$ is odd. Hence $n+1$ is even or odd, so $P(n+1)$ is true. If n is odd, then we may pick $k \in \mathbb{N}$ such that $n = 2k-1$, so $n+1 = 2k$, and then $n+1$ is even, so $n+1$ is even or odd, so $P(n+1)$ is odd.

So we have proved that $P(n+1)$ is true in both cases, so $P(n+1)$ is true.

Since we have proved $P(n+1)$ assuming $P(n)$, we have shown that $P(n) \Longrightarrow P(n+1)$. Since this has been established for arbitrary $n \in \mathbb{N}$, we have proved that $(\forall n \in \mathbb{N})(P(n) \Longrightarrow P(n+1))$. This concludes the inductive step.

Hence, by the PMI, $P(n)$ is true for all $n \in \mathbb{N}$. **Q.E.D.**

Theorem 2. *If n is a natural number and $n \neq 1$ then $n-1$ is a natural number.*

Proof. For $n \in \mathbb{N}$, let $P(n)$ be the statement "$n \neq 1 \Longrightarrow n-1 \in \mathbb{N}$".

We want to prove that $(\forall n \in \mathbb{N})P(n)$.

We will prove this by induction. For this purpose, we will prove that $P(1)$ and that $(\forall n)(P(n) \Longrightarrow P(n+1))$.

Base case. $P(1)$ is true, because $P(1)$ says that "$1 \neq 1 \Longrightarrow 1-1 \in \mathbb{N}$" and this is true because it is an implication whose premiss ("$1 \neq 1$") is false.

Inductive step. We must show that $(\forall n \in \mathbb{N})(P(n) \Longrightarrow P(n+1))$.

Let $n \in \mathbb{N}$ be arbitrary. We want to prove that $P(n) \Longrightarrow P(n+1)$. To show this, we assume $P(n)$ and prove $P(n+1)$.

So assume $P(n)$. That means that

(*) $n \neq 1 \Longrightarrow n-1 \in \mathbb{N}$.

Then $P(n+1)$ says

(#) $n+1 \neq 1 \Longrightarrow (n+1)-1 \in \mathbb{N}$.

But $(n+1)-1 = n$, and $n \in \mathbb{N}$, so the conclusion ("$(n+1)-1 \in \mathbb{N}$") of the implication (#) is true. Hence (#) is true. That is, $P(n+1)$ is true. So we have proved $P(n+1)$.

Since we have proved $P(n+1)$ assuming $P(n)$, we have shown that $P(n) \Longrightarrow P(n+1)$. Since this has been established for arbitrary $n \in \mathbb{N}$, we have proved that $(\forall n \in \mathbb{N})(P(n) \Longrightarrow P(n+1))$. This concludes the inductive step.

Hence, by the PMI, $P(n)$ is true for all $n \in \mathbb{N}$. **Q.E.D.**

Theorem 3. *If $n \in \mathbb{N}$ then there is no natural number q such that $n < q < n+1$.*

Proof. For $n \in \mathbb{N}$, let $P(n)$ be the statement "there is no natural number q such that $n < q < n+1$". We want to prove that $(\forall n \in \mathbb{N})P(n)$.

We will prove this by induction. For this purpose, we will prove that $P(1)$ and that $(\forall n)(P(n) \Longrightarrow P(n+1))$.

Base case. $P(1)$ is true, because $P(1)$ says that there are no natural numbers between 1 and 2, and this is true because of Theorem 2 of the previous lecture. ("Every natural number greater than 1 is greater than or equal to 2.")

Inductive step. We must show that $(\forall n \in \mathbb{N})(P(n) \Longrightarrow P(n+1))$.

Let $n \in \mathbb{N}$ be arbitrary. We want to prove that $P(n) \Longrightarrow P(n+1)$. To show this, we assume $P(n)$ and prove $P(n+1)$.

So assume $P(n)$. That means that

(*) There does not exist a natural number q such that $n < q < n+1$.

We want to prove that $P(n+1)$ is true, i.e.,

(#) There does not exist a natural number q such that $n+1 < q < n+2$.

We prove this by contradiction. Suppose that a natural number q such that $n+1 < q < n+2$ exists. Pick one such number and call it q_0. Then $q_0 \in \mathbb{N}$ and $n+1 < q_0 < n+2$. It then follows that $q_0 \neq 1$ (because $q_0 > n+1$ and $n+1 > 1$, so $q_0 > 1$).

So, by Theorem 2, $q_0 - 1$ is a natural number. And it is clear that $n < q_0 - 1 < n+1$. So there exists a natural number q such that $n < q < n+1$. But this contradicts (*).

So we have proved (#). Hence $P(n+1)$ is true.

Since we have proved $P(n+1)$ assuming $P(n)$, we have shown that $P(n) \Longrightarrow P(n+1)$. Since this has been established for arbitrary $n \in \mathbb{N}$, we have proved that $(\forall n \in \mathbb{N})(P(n) \Longrightarrow P(n+1))$. This concludes the inductive step.

Hence, by the PMI, $P(n)$ is true for all $n \in \mathbb{N}$. **Q.E.D.**

1.1 Inductive definitions

We have defined "x^2", for a real number x, to mean "$x.x$". And we can define "x^3" to mean "$(x.x).x$", or, if you prefer, "$x^2.x$". But how can we define "x^n" for an arbitrary natural number n? One possibility would be to write something like this

$$x^n = \underbrace{x \times x \times \cdots \times x}_{n \text{ times}}$$

But this is very unclear. I do not know what "$\cdots$" means, precisely (and if you think you do, please tell me!). And, in any case, "$\cdots$" is not one of the basic symbols that are all the symbols we are allowed to use. (That is, 0, 1, $+$, $-$, $\times$, $\div$, $=$, $<$, $\leq$, $\geq$, $>$, (,), $\forall$, $\exists$, $\wedge$, $\vee$, $\Longrightarrow$, $\Longleftrightarrow$, $\sim$, $\in$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, plus letter variables, and symbols defined later, such as 2, 3, 4, $|$, and $\subseteq$).

The way to define "x^n" correctly is by means of an inductive definition: we first define x^1 to be x, and then define x^{n+1} to be $x^n.x$, for every n. That is, we write:

Definition 1. ***(Inductive definition of positive integer powers of a real number)*** For all $a \in \mathbb{R}$, we set

$$\begin{aligned} a^1 &= a\,, \\ a^{n+1} &= a^n.a \quad \text{for } n \in \mathbb{N}\,. \end{aligned}$$

We also set $a^0 = 1$. □

Using this definition, we can write down what a^n is for any n.

Suppose, for example, that we want to know what a^5 is. By the second line of our inductive definition of a^n,

$$a^5 = a^4.a.$$

This answers our question about a^5, in terms of a^4. And what is a^4? Again, using the second line of the inductive definition, we find

$$a^4 = a^3.a.$$

So

$$a^5 = ((a^3).a).a.$$

And what is a^3? Once again, we can use the second line of the inductive definition, and find

$$a^3 = a^2.a$$

So

$$a^5 = (((a^2).a).a).a.$$

One more step yields

$$a^2 = a^1.a\,,$$

so

$$a^5 = (((a^1.a).a).a).a.$$

And, finally, the first line of the inductive definition, tells us that $a^1 = a$, so we end up with

$$a^5 = (((a.a).a).a).a.$$

Here are a few examples of inductive definitions:

Inductive definition of the factorial. The "factorial" of a natural number n is supposed to be the product $1 \times 2 \times 3 \times \cdots \times n$. That is, the factorial of n is the product of all the natural numbers from 1 to n. Here is the inductive definition:

Definition 2. The factorial of a natural number n is the number $n!$ given by

$$\begin{aligned} 1! &= 1\,, \\ (n+1)! &= n! \times (n+1) \quad \text{for } n \in \mathbb{N}\,. \end{aligned}$$

In addition, we define

$$0! = 1\,,$$

so $n!$ is defined for every nonnegative integer n.

Inductive definition of summation.

Definition 3. For a natural number n, and real numbers $a_1, a_2, \ldots, a_n$, we define the sum (or summation) of the a_j for j from 1 to n to be the number $\sum_{j=1}^{n} a_j$ determined as follows:

$$\begin{aligned} \sum_{j=1}^{1} a_j &= a_1\,, \\ \sum_{j=1}^{n+1} a_j &= \Big(\sum_{j=1}^{n} a_j\Big) + a_{n+1} \quad \text{for} \quad n \in \mathbb{N}\,. \end{aligned}$$

And we also define $\sum_{j=1}^{0} a_j = 0$.

Inductive definition of product.

Definition 4. For a natural number n, and real numbers $a_1, a_2, \ldots, a_n$, we define the product of the a_j for j from 1 to n to be the number $\prod_{j=1}^{n} a_j$ determined as follows:

$$\begin{aligned} \prod_{j=1}^{1} a_j &= a_1\,, \\ \prod_{j=1}^{n+1} a_j &= \Big(\prod_{j=1}^{n} a_j\Big) \times a_{n+1} \quad \text{for} \quad n \in \mathbb{N}\,. \end{aligned}$$

And we also define $\prod_{j=1}^{0} a_j = 1$.

1.1.1 Some examples of proofs using inductive definitions

Theorem 4. *For all $n \in \mathbb{N}$*

$$\sum_{k=1}^{n}(2k-1) = n^2.$$

(That is, the sum of the first n odd numbers is a perfect square, namely, n^2.)

Proof. For $n \in \mathbb{N}$, let $P(n)$ be the statement "$\sum_{k=1}^{n}(2k-1) = n^2$ ". We want to prove that $(\forall n \in \mathbb{N})P(n)$.

We will prove this by induction. For this purpose, we will prove that $P(1)$ and that $(\forall n)(P(n) \Longrightarrow P(n+1))$.

Base case. $P(1)$ is true, because $P(1)$ says that $\sum_{k=1}^{1}(2k-1) = 1^2$, which is true because $\sum_{k=1}^{1}(2k-1) = 1$ and $1^2 - 1$.

Inductive step. We prove that $(\forall n \in \mathbb{N})(P(n) \Longrightarrow P(n+1))$.

Let n ne an arbitrary natural number. We want to prove that $P(n) \Longrightarrow P(n+1)$. To prove this, we assume $P(n)$ and prove $P(n+1)$.

So assume that $P(n)$ is true. Then

$$\sum_{k=1}^{n}(2k-1) = n^2\,.$$

We want to prove that $P(n+1)$ is true, i.e., that

$$\sum_{k=1}^{n+1}(2k-1) = (n+1)^2\,. \tag{1.1}$$

But, by the inductive definition of "summation",

$$\sum_{k=1}^{n+1}(2k-1) = \sum_{k=1}^{n}(2k-1) + (2(n+1)-1)\,.$$

And our inductive assumption tells us that

$$\sum_{k=1}^{n}(2k-1) = n^2\,.$$

So

$$\begin{aligned}\sum_{k=1}^{n+1}(2k-1) &= \sum_{k=1}^{n}(2k-1)+(2(n+1)-1)\\ &= n^2+(2(n+1)-1)\\ &= n^2+(2n+2-1)\\ &= n^2+2n+1\\ &= (n+1)^2\,.\end{aligned}$$

Hence (1.1) is true. So we have proved that $P(n+1)$ is true, assuming $P(n)$.

Since we have proved $P(n+1)$ assuming $P(n)$, we have shown that $P(n) \Longrightarrow P(n+1)$. Since this has been established for arbitrary $n \in \mathbb{N}$, we have proved that $(\forall n \in \mathbb{N})(P(n) \Longrightarrow P(n+1))$. This concludes the inductive step.

Hence, by the PMI, $P(n)$ is true for all $n \in \mathbb{N}$. **Q.E.D.**

Theorem 5. *For all* $n \in \mathbb{N}$

$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\,.$$

Proof. For $n \in \mathbb{N}$, let $P(n)$ be the statement "$\sum_{k=1}^{n} k^2 = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$". We want to prove that $(\forall n \in \mathbb{N})P(n)$.

We will prove this by induction. For this purpose, we will prove that $P(1)$ and that $(\forall n)(P(n) \Longrightarrow P(n+1))$.

Base case. $P(1)$ is true, because $P(1)$ says that

$$\sum_{k=1}^{1} k^2 = \frac{1}{3}+\frac{1}{2}+\frac{1}{6}\,,$$

and this is true because $\sum_{k=1}^{1} k^2 = 1$ and $\frac{1}{3}+\frac{1}{2}+\frac{1}{6} = 1$.

Inductive step. We prove that $(\forall n \in \mathbb{N})(P(n) \Longrightarrow P(n+1))$.

Let n ne an arbitrary natural number. We want to prove that $P(n) \Longrightarrow P(n+1)$. To prove this, we assume $P(n)$ and prove $P(n+1)$.

So assume that $P(n)$ is true. Then

$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\,.$$

We want to prove that $P(n+1)$ is true, i.e., that

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)^3}{3} + \frac{(n+1)^2}{2} + \frac{n+1}{6}\,. \tag{1.2}$$

But, by the inductive definition of "summation",

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2\,.$$

And our inductive assumption tells us that

$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\,.$$

So

$$\begin{aligned} \sum_{k=1}^{n+1} k^2 &= \Big(\sum_{k=1}^{n} k^2\Big) + (n+1)^2 \\ &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} + (n+1)^2\,. \end{aligned}$$

Now we have to prove that

$$\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} + (n+1)^2 = \frac{(n+1)^3}{3} + \frac{(n+1)^2}{2} + \frac{n+1}{6}\,. \tag{1.3}$$

But

$$(n+1)^2 = n^2 + 2n + 1\,,$$

so

$$\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} + (n+1)^2 = \frac{n^3}{3} + \frac{3n^2}{2} + \frac{13n}{6} + 1\,,$$

and

$$\begin{aligned} \frac{(n+1)^3}{3} + \frac{(n+1)^2}{2} + \frac{n+1}{6} &= \frac{n^3+3n^2+3n+1}{3} + \frac{n^2+2n+1}{2} + \frac{n+1}{6} \\ &= \frac{n^3}{3} + n^2 + n + \frac{1}{3} + \frac{n^2}{2} + n + \frac{1}{2} + \frac{n}{6} + \frac{1}{6} \\ &= \frac{n^3}{3} + \frac{3n^2}{2} + \frac{13n}{6} + 1\,. \end{aligned}$$

So both sides of (1.3) are equal to $\frac{n^3}{3} + \frac{3n^2}{2} + \frac{13n}{6} + 1$. Hence (1.3) is true, and therefore (1.2) is true. Sp $P(n+1)$ is true. So we have proved that $P(n+1)$ is true, assuming $P(n)$.

Since we have proved $P(n+1)$ assuming $P(n)$, we have shown that $P(n) \implies P(n+1)$. Since this has been established for arbitrary $n \in \mathbb{N}$, we have proved that $(\forall n \in \mathbb{N})(P(n) \implies P(n+1))$. This concludes the inductive step.

Hence, by the PMI, $P(n)$ is true for all $n \in \mathbb{N}$. **Q.E.D.**