MATHEMATICS H311 — FALL 2015

Introduction to Mathematical Analysis

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NOTES ON THE FINAL EXAM

Dear students:

Here are some last-minute remarks that may be of some use to you for the final exam. **Read them carefully.** Also, I include a couple of examples of problems pertaining to the last couple of lectures, with solutions, which are similar to problems you may encounter in the final exam.

Some of these remarks are based mainly on what I saw in the second midterm, and in particular on mistakes many students that I believe you should be aware of in order not to repeat them.

1. The exam is going to have a format similar to that of the two midterms, except for the fact that there will be ***two parts***, worth 100 points each:

   (a) Part I will consist of problems, adding up to 100 points, that everybody will have to do, with no margin for choice;

   (b) Part II will also be worth 100 points, but will consist a set of problems with point values adding up to approximately 165, so you will be able to choose and you will not have to do all the problems.

2. Every problem in the final exam will be of one of the following three kinds:

   a. a homework problem (and by this I mean ANY problem that appeared in the list of homework problems, not only those that were labelled "to hand in");

   b. a problem that appeared in one of the two midterms or quizzes,

   c. a problem about the very last couple of lectures, about which I have not had the occasion to assign homework problems. I am including below some examples of possible problems of this kind.

3. It seems to me that many students do not know yet how to write correct definitions and statements of theorems. ***You should make sure you know how to do that.*** This involves, in particular,

   (a) correct use of logical connectives and quantifiers, as you learned in Math 300; for example, several students got the definition of "continuous function" wrong because they put the quantifiers in the wrong order.

   For example, some students wrote something more or less like this[1]:

   (#) Let X, Y be metric spaces, with distance functions d_X, d_Y, and let $f : X \mapsto Y$ be a function. Let $(x_n)_{n=1}^{\infty}$ be a sequence of points of X, and let $x = \lim_{n\to\infty} x_n$. The function f is continuous if $\lim_{n\to\infty} f(x_n) = f(x)$.

   This is ***completely wrong***. In ordee for you to understand why, I recommend that you do the following

   Exercise 1. Prove that, according to Definition (#), *every* function is continuous. □

   The correct definition of "continuous function" is totally different. It says

   (∗) Let X, Y be metric spaces, with distance functions d_X, d_Y, and let $f : X \mapsto Y$ be a function. We say that the function f is continuous if for *every* sequence $(x_n)_{n=1}^{\infty}$ of points of X, and *every* point x of X, if $\lim_{n\to\infty} x_n = x$ then $\lim_{n\to\infty} f(x_n) = f(x)$.

   See how different this is from (#)? In the correct definition ***there are universal quantifiers and an implication!!!***.

[1] This atrocity is a composite of what several students wrote. Nobody wrote exactly that, but several people wrote things similar to it, and equally bad.

So here is a correct definition of "comtinuous function":

> If X and Y be metric spaces, with distance functions, d_X, d_Y, and $f : X \mapsto Y$ is a function, then f is continuous if
>
> $$(\forall \mathbf{x})(\forall x)\Bigg(\Big(\mathbf{x} = (x_n)_{n=1}^{\infty} \wedge (\forall n \in \mathbb{N}) x_n \in X \wedge x \in X \wedge \lim_{n\to\infty} x_n = x\Big) \Longrightarrow \lim_{n\to\infty} f(x_n) = f(x)\Bigg).$$

And if you miss the quantifiers or the implication, or put them in the wrong place, then you get a totally different definition, and unless a miracle happens this definition of yours is going to be wrong!!!

And, to conclude, here are some problems on derivatives that may appear in the final exam:

1. **Prove** that if I is an interval, $f : I \mapsto \mathbb{R}$ is a function, f is differentiable at every point of I, and $f'(x) = 0$ for every $x \in I$, then f is a constant function. *(This is Corollary 5.3.3 in the book.)*

2. If I is an interval, a function $f : I \mapsto \mathbb{R}$ is Lipschitz if there exists a real number C such that

$$|f(x) - f(y)| \leq C|x - y| \qquad \text{for every} \quad x, y \in I.$$

Prove that if f is differentiable at every $x \in I$, and the derivative f' is bounded (that is, there exists a constant B such that $|f'(x)| \leq B$ for every $x \in I$) then f is Lipschitz.

3. Book, Exercises 5.3.2, 5.3.3, 5.3.4, on Page 161.

4. State and prove the Differentiable Limit Theorem. (This is Theorem 6.3.1 in the book. I gave a detailed proof in class, assuming that the functions f_n and g are continuous. You are allowed to make this assumption as well if you want to. You can give the proof I gave in class or the proof in the book, whichever you like best.)

5. Exercises 6.3.1, 6.3.2, 6.3.3 on pages 186, 187 of the book.