MATHEMATICS 300 — FALL 2009

Introduction to Mathematical Reasoning

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INSTRUCTOR'S NOTES

(October 15, 2009)

1 Homework assignment No. 7, due on Tuesday October 27.

1. Prove that $\sqrt{6}$ is irrational.
2. Prove that $\sqrt[3]{2}$ is irrational.
3. Prove that $\sqrt{2}+\sqrt[3]{2}$ is irrational.
4. Prove that if $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are real numbers then
$$a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 \le \sqrt{a_1^2+a_2^2+a_3^2+a_4^2}\sqrt{b_1^2+b_2^2+b_3^2+b_4^2}\,.$$
5. Prove that if $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5$ are real numbers then
$$\begin{aligned}&a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 + a_5b_5 \le \\ &\qquad\sqrt{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2}\sqrt{b_1^2+b_2^2+b_3^2+b_4^2+b_5^2}\,.\end{aligned}$$

NOTE: In an earlier assignment, you had to prove that if a, b, c, d are real numbers, then
$$ab + cd \le \sqrt{a^2+c^2}\sqrt{b^2+d^2}\,,$$
that is, that if a_1, a_2, b_1, b_2 are real numbers, then
$$a_1b_1 + a_2b_2 \le \sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_2^2}\,.$$
Later, you were asked to prove the same thing for two groups a_1, a_2, a_3, b_1, b_2, b_3 of three numbers; that is, you had to prove that if $a_1, a_2, a_3, b_1, b_2, b_3$ are real numbers then
$$a_1b_1 + a_2b_2 + a_3b_3 \le \sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}\,.$$
Now I am asking you to prove the same thing for two groups of four numbers, and for two groups of five numbers. If you think that the formulas are getting too long and this is getting repetitive, if you think that Problems 4 and 5 above are similar to problems you did earlier, see if you can figure out a way to prove, once and for all, the following statement

(#) If n is any natural number, and $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are real numbers, then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \leq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}\sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}\,.$$

If you can prove (#), then the four problems with groups of two, three, four, and five numbers just become special cases of a general inequality, so you know equally well that, for example, if

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}$$

are real numbers then

$$\begin{aligned} a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 + a_5b_5 + a_6b_6 + a_7b_7 + a_8b_8 + a_9b_9 + a_{10}b_{10} \leq \\ \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2 + a_9^2 + a_{10}^2} \\ \times\sqrt{b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + b_6^2 + b_7^2 + b_8^2 + b_9^2 + b_{10}^2}\,. \end{aligned}$$

and you do not have to give a separate proof of this with very long formulas.