MATHEMATICS 300 — FALL 2009
Introduction to Mathematical Reasoning
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INSTRUCTOR'S NOTES

(November 14, 2009)

1 Homework assignment No. 11, due on Thursday, November 19, 2009

This homework appears long, but that's just because it has lots of explanations and hints. But it really is rather short, certainly a lot shorter than the previous assignment.

1. A subset S of $\mathbb{N}$ is **inductive** if

(a) $1 \in S$,

(b) $(\forall n \in \mathbb{N})(n \in S \Rightarrow n+1 \in S)$.

Prove that if $S \subseteq \mathbb{N}$ then S is inductive if and only if $S = \mathbb{N}$. (NOTE: this is basically done in the book, but I want you to write down the proof, because it is important. To prove that $S = \mathbb{N}$ if S is inductive, prove by induction that "$n \in S$" is true for all $n \in \mathbb{N}$, that is, that $(\forall n \in \mathbb{N}) n \in S$.)

IMPORTANT REMARK. The result of this problem gives rise to an alternative way to write proofs by induction. Instead of singling out the statement $P(n)$ that we want to prove to be true for all $n \in \mathbb{N}$, we introduce the set of all $n \in \mathbb{N}$ for which the desired statement is true, and prove that S is inductive, from which we can conclude that $S = \mathbb{N}$. For example, to prove that

$$(\forall n \in \mathbb{N}) \sum_{k=1}^{n} k = \frac{n(n+1)}{2}\,, \qquad (1)$$

we could proceed as follows: Let S be the set of all $n \in \mathbb{N}$ such that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. We will prove that S is inductive. First we have to show that $1 \in S$, but this follows easily because if $n = 1$ then $\sum_{k=1}^{n} k = 1$ and $\frac{n(n+1)}{2} = 1$. Next we carry out the inductive step, by proving that $(\forall n \in \mathbb{N})(n \in S \Rightarrow n+1 \in S)$. Let $n \in \mathbb{N}$ be arbitrary. Assume that $n \in S$. Then $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. Adding $n+1$ to both sides, we get

$$\begin{aligned}\sum_{k=1}^{n+1} k &= \Big(\sum_{k=1}^{n} k\Big) + n + 1 \\ &= \frac{n(n+1)}{2} + n + 1 \\ &= (n+1)\Big(1 + \frac{n}{2}\Big) \\ &= (n+1)\frac{n+2}{2} \\ &= \frac{(n+1)(n+2)}{2} \\ &= \frac{(n+1)((n+1)+1)}{2},\end{aligned}$$

so $n+1 \in S$. So we have shown that S is inductive. Hence $S = \mathbb{N}$, which means precisely that (1) holds.

2. A subset S of $\mathbb{Z}$ is **biinductive** if

(a) $0 \in S$,

(b) $(\forall n \in \mathbb{Z})(n \in S \Rightarrow n+1 \in S)$.

(c) $(\forall n \in \mathbb{Z})(n \in S \Rightarrow n-1 \in S)$.

Prove that if $S \subseteq \mathbb{Z}$ then S is biinductive if and only if $S = \mathbb{Z}$. (HINT: To prove that $S = \mathbb{Z}$ if S is biinductive, prove by induction that "$n \in S$" is true for all $n \in \mathbb{N}$ and "$-n \in S$" is true for all $n \in \mathbb{N}$.)

3. Using the result of Problem 2, **prove** that every integer is even or odd, that is, that

every integer is of the form $2q$ or $2q+1$, for some integer q,

or, in formal language,

$$(\forall n \in \mathbb{Z})(\exists q \in \mathbb{Z})(n = 2q \vee n = 2q+1)\,.$$

(HINT: Let S be the set of all $n \in \mathbb{Z}$ having the property that $(\exists q \in \mathbb{Z})(n = 2q \vee n = 2q+1)$. Prove that S is biinductive. Notice that, in the "biinductive step," to go from "$n \in S$" to "$n+1 \in S$" or to "$n-1 \in S$", q may change. So, for example, $7 = 2q+1$, with $q = 3$, but when you go up from 7 to 8, you get $8 = 2q + 1 + 1 = 2(q + 1)$, so the new q is $q + 1$.)

4. Using the result of Problem 2, **prove** that

every integer is of the form $3q$, or $3q + 1$, or $3q + 2$, for some integer q,

that is, that

$$(\forall n \in \mathbb{Z})(\exists q \in \mathbb{Z})\Big(n = 3q \vee n = 3q + 1 \vee n = 3q + 2\Big) .$$

(HINT: Let S be the set of all $n \in \mathbb{Z}$ having the property that $(\exists q \in \mathbb{Z})(n = 3q \vee n = 3q + 1 \vee n = 3q + 2)$. Prove that S is biinductive. Again, as in Problem 2, you have to worry about "updating" the number q. For example, $6 = 3q$ with $q = 2$; when you go from 6 to 7 you get $7 = 3q + 1$, with the same q; and when you go from 7 to 8 you get $8 = 3q + 2$, with the same q; but when you go from 8 to 9 you get $9 = 3q + 2 + 1$, i.e., $9 = 3(q + 1)$, so the new q is $q + 1$. What you are doing here is "counting modulo 3": the number r such that $n = 3q + r$ goes up each time by 1, but when it gets to 3 you reset it to zero and raise q by 1. So r always remains ≥ 0 and < 3.)

5. Generalize the results of Problem 3 (which was the Division Theorem for $b = 2$) and Problem 4 (which was the Division Theorem for $b = 3$) and **prove**, using the result of Problem 2, the *Division Theorem for dividing by any positive integer* b:

given any positive integer b, and any integer a, we can write $a = bq + r$ for some integer q and some integer r such that r is one of the numbers $0, 1, \ldots, b - 1$.

That is,

$$(\forall b \in \mathbb{N})(\forall a \in \mathbb{Z})(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})\Big(a = bq + r \wedge 0 \leq r < b\Big) .$$

NOTE: We proved this in class, using well-ordering. Here I am asking you to prove it using "biinduction", that is, the result of Problem 2. You should first fix b ("let $b \in \mathbb{N}$ be arbitrary"), and then, for that fixed b, you should consider the set A of all integers a having the property that $(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})\Big(a = bq + r \wedge 0 \leq r < b\Big)$. Then you should prove that A is biinductive.

6 Generalize the result of Problem 5 and **prove**, using the result of Problem 5, the *Division Theorem for dividing by any nonzero integer* b:

given any nonzero integer b, and any integer a, we can write $a = bq + r$ for some integer q and some integer r such that $r = 0 \vee r = 1 \vee \cdots \vee r = \|b\| - 1$.

That is,

$$(\forall b \in \mathbb{Z})\Big(b \neq 0 \Longrightarrow (\forall a \in \mathbb{Z})(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})\big(a = bq + r \wedge 0 \leq r < |b|\big)\Big).$$

NOTE: This is very easy and very short. You do not need to do a proof by induction or by biinduction. Just use the result of Problem 5 and you will get a two-line proof. (Well, maybe three lines.)

7. (i) Verify the identity

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2. \tag{2}$$

for $n = 3$, $n = 4$, and $n = 5$. (Use Pascal's triangle[1] to compute the binomial coefficients you need.)

(ii) Prove that (2) holds for all all $n \in \mathbb{N}$. (HINT: Do *not* try to use induction, because that would be a dead end. Use a combinatorial proof: observe that $\binom{2n}{n}$ is the number of subsets with n members of a set S with $2n$ members; then try to count the number of these subsets in a different way: divide up the set S into two sets S_g—the "green" members of S—and S_r—the "red" members of S, each one having n members; then observe that to form a subset of S with n members you could pick, for example, 3 green members and $n - 3$ red ones, or 5 green members and $n - 5$ red ones, or,

[1] **Pascal's triangle** is the table of binomial coefficients that I wrote on the blackboard; it starts with:

$n = 0:$	1			
$n = 1:$	1	1		
$n = 2:$	1	2	1	
$n = 3:$	1	3	3	1

and in class I wrote the triangle for all the binomial coefficients $\binom{n}{k}$ for n up to 11.

in general, k green members and $n-k$ red ones, for any $k \in \mathbb{Z}$ such that $0 \leq k \leq n$. Also, you will find the identity $\binom{n}{n-k} = \binom{n}{k}$ useful.)