# MATHEMATICS 300 — FALL 2009

*Introduction to Mathematical Reasoning*

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**INSTRUCTOR'S NOTES**

*(November 22, 2009)*

## 1 Homework assignment No. 12, due on Tuesday, December 1, 2009

**1.** Prove (by induction) that, if $f$ and $g$ are two functions defined on an interval $I$ and having derivatives of all orders (that is, such that, for every $n \in \mathbb{N}$, the $n$-th derivatives $f^{(n)}(x)$ and $g^{(n)}(x)$ exist for every $x \in I$) then

$$(\forall n \in \mathbb{N})(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)} \,. \qquad (1)$$

(Note: the inductive definition of $h^{(m)}$, for a function $h$ and a nonnegative integer $m \in \mathbb{N} \cup \{0\}$, is as follows:

$$\begin{aligned} h^{(0)} &= h \,, \\ h^{(m+1)} &= \frac{dh^{(m)}}{dx} \qquad \text{for} \quad m \in \mathbb{N} \cup \{0\} \,. \end{aligned}$$

One usually writes $h'$ for $h^{(1)}$, $h''$ for $h^{(2)}$, and $h'''$ for $h^{(3)}$.)

HINT: Use the same method as in the proof of the binomial theorem that we did in class. Use the Leibniz formula

$$(fg)' = fg' + f'g$$

to get started, and also in the inductive step.

**1a.** (OPTIONAL HARD PROBLEM) If you did Problem 1 following the hint, you will have noticed that the proof of Equation (1) and the proof of the binomial theorem are "almost exactly the same proof." See if you can figure out a way to subsume both results (the binomial theorm and Formula (1)) in one single result, so you do not need to do the proof twice.

**2.** Book, Page 117, Problem 11.

**3.** Book, page 128, Problem 18(a). HINT: You may find the following trivial observation useful: if $A$ is a set with $n$ members, and $n$ is odd, then for every subset $S$ of $A$, $S$ has an even number of members if and only if $A - S$ has an odd number of members.

*For an alternative way to do this problem, see the remark after Problem 4.*

**4.** Prove the result of the previous problem for $n$ even and greater than 0. (Also, show that for $n = 0$ the result is false.) The strategy suggested in the hint of Problem 3 will not work here, but a more complicated combinatorial proof, counting sets, will work. I recommend that you first do it for some low values of $n$—for example, for $n = 4$—to see what is happening. For example, you will notice things such as this: if you let $p$ be the number of subsets of a set $\{a, b, c, d\}$ of four elements that contain $a$ and have an even number of members, and let $q$ be the number of subsets that do not contain $a$ and have an odd number of members, then $p = q$. (Indeed, the sets that contain $a$ and have an even number of members are $\{a, b\}, \{a, c\}, \{a, d\}$, and $\{a, b, c, d\}$, so $p = 4$. The sets that do not contain $a$ and have an odd number of members are $\{b\}, \{c\}, \{d\}$, and $\{b, c, d\}$, so $q = 4$.) Also, if you let $r$ be the number of subsets of $\{a, b, c, d\}$ that do not contain $a$ and have an even number of members, and let $s$ be the number of subsets that contain $a$ and have an odd number of members, then $p = q$. (Indeed, the sets that do not contain $a$ and have an even number of members are $\emptyset, \{b, c\}, \{b, d\}$ and $\{c, d\}$, so $r = 4$. The sets that contain $a$ and have an odd number of members are $\{a\}, \{a, b, c\}, \{a, b, d\}$, and $\{a, c, d\}$, so $s = 4$.) The total number of sets with an even number of members is $p + r$ ("even with $a$ plus even without $a$"), and the total number of sets with an odd number of members is $q + s$ ("odd without $a$ plus odd with $a$"). Since $p = q$ and $r = s$, both total numbers—the number of subsets with an even number of members, and the number of subsets with an odd number of members—are the same.

REMARK: As an alternative, if you do not wish to do a combinatorial proof, you may do an algebraic proof, using the binomial theorem with $a = 1$ and $b = -1$. (This works both for $n$ odd and $n$ even and $> 0$, but of course it shouldn't work, and it doesn't work, for $n = 0$.) If you choose this strategy, I also recommend that you first do it for some low values of $n$—for example, for $n = 5$ and $n = 6$—to see what is happening.