MATHEMATICS 300 — FALL 2009

Introduction to Mathematical Reasoning

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INSTRUCTOR'S NOTES

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1 How sentences can be combined to form new sentences

Sentences can be combined to form new sentences by means of the **logical connectives**. There are exactly seven of them, listed in the box.

THE SEVEN LOGICAL CONNECTIVES

symbol	name	meaning
$\sim$	negation	not, it's not the case that
$\vee$	disjunction	or
$\wedge$	conjunction	and
$\Rightarrow$	implication	implies, if . . . then
$\Leftrightarrow$	biconditional	if and only if
$\forall$	universal quantifier	for all
$\exists$	existential quantifier	there exists . . . such that

The *negation* connective is applied to *one* sentence to form another sentence, called the *negation* of the first sentence. For example, the sentence "$\sim x = 3$" is the negation of "$x = 3$", and is read "it's not the case that x equals three," or "x is not equal to three".

The *disjunction* connective is used to combine *two* sentences to form another sentence, called the *disjunction* of the first two sentences. For example, the sentence "$x = 3 \vee x > 2$" is the disjunction of the sentences "$x = 3$" and "$x > 2$", and is read "x equals three or x is greater than two".

The *conjunction* connective is used to combine *two* sentences to form another sentence, called the *conjunction* of the first two sentences. For example, the sentence "$x = 3 \wedge x > 2$" is the conjunction of the sentences "$x = 3$" and "$x > 2$", and is read "x equals three and x is greater than two".

The *implication* connective is used to combine *two* sentences to form another sentence, called an *implication.* More precisely, if A and B are sentences, then $A \Rightarrow B$ is a sentence, said to be the *implication with premiss A and conclusion B*. For example, the sentence "$x = 3 \Rightarrow x > 2$" is the implication with premiss "$x = 3$" and conclusion "$x > 2$". (NOTE: another word for "premiss" is "antecedent"; also, another word for "conclusion" is "consequent".) We read "$x = 3 \Rightarrow x > 2$" as "$x = 3$ implies $x > 2$", or "if $x = 3$ then $x > 2$". **You should not read "$x = 3 \Rightarrow x > 2$" as "$x = 3$ therefore $x > 2$". The sentence "$x = 3 \Rightarrow x > 2$" does not make the assertion that $x = 3$. (For example, "$x = 3 \Rightarrow x > 2$" is true if $x = 6$.)**

The *biconditional* connective is used to combine *two* sentences to form another sentence, called an *equivalence*, or *biconditional.* For example, the sentence "$x = 3 \Leftrightarrow x > 2$" is an equivalence. We read "$x = 3 \Leftrightarrow x > 2$" as "$x = 3$ if and only if $x > 2$".

The *universal quantifier* connective is applied to *one* sentence to form another sentence, called a *universal sentence.* For example, from the sentence "$x = 3 \vee x > 2$" we can get the universal sentence

$$(\forall x)(x = 3 \vee x > 2)\,.$$

This is read as "for all x, x equals three or x is greater than two", or "every x is either equal to three or greater than two".

Usually, in a universally quantified sentence such as the one above, the whole expression $(\forall x)$ is referred to as a *quantifier.*

Universal quantifiers can be restricted, by specifying a range for the variable x. For example, the sentence

$$(\forall x \in \mathbb{R})(x = 3 \vee x > 2)$$

says that "for all real numbers x, x equals three or x is greater than two", or "every real number is either equal to three or greater than two".

The sentence to which the universal quantifier applies is the *scope* of the quantifier. *The scope of a quantifier must be enclosed in parenthesis.* If there is no left parenthesis immediately to the right of a quantifier $(\forall x)$ or $(\forall x \in U)$, then the scope of the quantifier is taken to be the smallest sentence to the right of the quantifier.

For example, let us compare the sentences

$$(1) \qquad (\forall x \in \mathbb{R})(x = 3 \vee x > 2)\,.$$

and

$$(2) \qquad (\forall x \in \mathbb{R})x = 3 \vee x > 2\,.$$

In (1), the scope of the quantifier is "$x = 3 \vee x > 2$", so the sentence is read as "for all real numbers x, x equals three or x is greater than two". In (2), the scope of the quantifier is "$x = 3$", so the sentence is read as "either all real numbers are equal to three, or x is greater than two". Notice that **misplaced parentheses or missing parentheses can make a huge difference.**

The *existential quantifier* connective is applied to *one* sentence to form another sentence, called an *existential sentence*. For example, from the sentence "$x = 3 \vee x > 2$" we can get the existential sentence

$$(\exists x)(x = 3 \vee x > 2)\,.$$

This is read as "there exists x such that either x equals three or x is greater than two", or "some x is either equal to three or greater than two".

Usually, in an existentially quantified sentence such as the one above, the whole expression $(\exists x)$ is referred to as a *quantifier*.

Existential quantifiers can be restricted, by specifying a range for the variable x. For example, the sentence

$$(\exists x \in \mathbb{R})(x = 3 \vee x > 2)$$

says that "there exists a real number x such that x equals three or x is greater than two", or "some real number is either equal to three or greater than two".

The sentence to which the existential quantifier applies is the *scope* of the quantifier. *The scope of a quantifier must be enclosed in parenthesis.* If there is no left parenthesis immediately to the right of a quantifier $(\exists x)$ or $(\exists x \in U)$, then the scope of the quantifier is taken to be the smallest sentence to the right of the quantifier.

For example, let us compare the sentences

$$(3) \qquad (\exists x \in \mathbb{R})(x = 3 \vee x > 2)\,.$$

and

$$(4) \qquad (\exists x \in \mathbb{R})x = 3 \vee x > 2\,.$$

In (3), the scope of the quantifier is "$x = 3 \vee x > 2$", so the sentence is read as "there exists a real number x such that either x equals three or x

is greater than two". In (4), the scope of the quantifier is "$x = 3$", so the sentence is read as "either there exists a real number which is equal to three, or x is greater than two". Once again, you should notice that **misplaced parentheses or missing parentheses can make a huge difference.**

2 Homework assignment No. 2, due on Thursday Sept. 17

Book, pages 26-27-28, problems 1 (non-starred items), 3, 5 (non-starred items), 6 (non-starred items), 8 (non-starred items), 10 (non-starred items), 11.