

MATHEMATICS 300 — SPRING 2010

Introduction to Mathematical Reasoning

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INSTRUCTOR'S NOTES

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1 How sentences can be combined to form new sentences

Sentences can be combined to form new sentences by means of the **logical connectives**. There are exactly seven of them, listed in the box.

THE SEVEN LOGICAL CONNECTIVES		
symbol	name	meaning
\sim	negation	not, it's not the case that
\vee	disjunction	or
\wedge	conjunction	and
\Rightarrow	implication	implies, if...then
\Leftrightarrow	biconditional	if and only if
\forall	universal quantifier	for all
\exists	existential quantifier	there exists...such that

Negation (“not”). The *negation* connective is applied to *one* sentence to form another sentence, called the *negation* of the first sentence. (So we say that *negation is a one-argument connective*.)

For example, the sentence “ $\sim x = 3$ ” is the negation of “ $x = 3$ ”, and is read “it’s not the case that x equals three,” or “ x is not equal to three”.

Disjunction (“or”). The *disjunction* connective is used to combine *two* sentences P and Q to form another sentence $P \vee Q$, called the *disjunction* of P and Q . (So we say that *disjunction is a two-argument connective*.) For example, the sentence “ $x = 3 \vee x > 2$ ” is the disjunction of the sentences “ $x = 3$ ” and “ $x > 2$ ”, and is read “ x equals three or x is greater than two”. In a disjunction $P \vee Q$ of two sentences P , Q , these two sentences are the *disjuncts*.

Conjunction (“and”). The *conjunction* connective is a two-argument connective: it is used to combine *two* sentences P and Q to form another sentence $P \wedge Q$, called the *conjunction* of P and Q . For example, the sentence “ $x = 3 \wedge x > 2$ ” is the conjunction of the sentences “ $x = 3$ ” and “ $x > 2$ ”, and is read “ x equals three and x is greater than two”. In a conjunction $P \vee Q$ of two sentences P, Q , these two sentences are the *conjuncts*.

Implication (“implies”, “If ... then ...”, “only if”). The *implication* (or *conditional*) connective is a two-argument connective: it is used to combine *two* sentences P and Q to form another sentence $P \Rightarrow Q$, called an *implication*. In an implication $P \Rightarrow Q$, P is the *premiss* (or *antecedent*) and Q is the *conclusion* (or *consequent*). For example, the sentence “ $x = 3 \Rightarrow x > 2$ ” is the implication with premiss “ $x = 3$ ” and conclusion “ $x > 2$ ”. We read “ $x = 3 \Rightarrow x > 2$ ” as

$x = 3$ implies $x > 2$,

or if $x = 3$ then $x > 2$,

or $x > 2$ if $x = 3$,

or $x = 3$ only if $x > 2$.

You should not read the sentence “ $x = 3 \Rightarrow x > 2$ ” as “ $x = 3$ therefore $x > 2$ ”. The sentence “ $x = 3 \Rightarrow x > 2$ ” does not make the assertion that $x = 3$. (For example, “ $x = 3 \Rightarrow x > 2$ ” is true if $x = 6$.)

Biconditional (“if and only if”). The *biconditional* connective is a two-argument connective: it is used to combine *two* sentences to form another sentence, called an *equivalence*, or *biconditional*. For example, the sentence “ $x = 3 \Leftrightarrow x > 2$ ” is an equivalence. We read “ $x = 3 \Leftrightarrow x > 2$ ” as “ $x = 3$ if and only if $x > 2$ ”.

Universal quantifiers (“for all”). *Universal quantifiers* are one-argument connectives consisting of a left parenthesis followed by the symbol \forall , followed by a letter variable such as x or y or a, b or ε , followed by a right parenthesis. Universal quantifiers are applied to a sentence to form another sentence, called a *universal sentence*. For example, from the sentence “ $x = 3 \vee x > 2$ ” we can get the universal sentence

$$(\forall x)(x = 3 \vee x > 2).$$

This is read as “for all x , x equals three or x is greater than two”, or “every x is either equal to three or greater than two”.

Universal quantifiers can be *restricted*, by specifying a range for the variable x . For example, the sentence

$$(\forall x \in \mathbb{R})(x = 3 \vee x > 2)$$

says that “for all real numbers x , x equals three or x is greater than two”, or “every real number is either equal to three or greater than two”.

The sentence to which the universal quantifier applies is the *scope* of the quantifier. *The scope of a quantifier must be enclosed in parenthesis.* If there is no left parenthesis immediately to the right of a quantifier ($\forall x$) or ($\forall x \in U$), then the scope of the quantifier is taken to be the smallest sentence to the right of the quantifier.

For example, let us compare the sentences

$$(1) \quad (\forall x \in \mathbb{R})(x = 3 \vee x > 2).$$

and

$$(2) \quad (\forall x \in \mathbb{R})x = 3 \vee x > 2.$$

In (1), the scope of the quantifier is “ $x = 3 \vee x > 2$ ”, so the sentence is read as “for all real numbers x , x equals three or x is greater than two”. In (2), the scope of the quantifier is “ $x = 3$ ”, so the sentence is read as “either all real numbers are equal to three, or x is greater than two”. Notice that **misplaced parentheses or missing parentheses can make a huge difference.**

Existential quantifiers (“for some”). *Existential quantifiers* are one-argument connectives consisting of a left parenthesis followed by the symbol \exists , followed by a letter variable such as x or y or a , b or ε , followed by a right parenthesis. Universal quantifiers are applied to a sentence to form another sentence, called an *existential sentence*. For example, from the sentence “ $x = 3 \vee x > 2$ ” we can get the existential sentence

$$(\exists x)(x = 3 \vee x > 2).$$

This is read as “there exists x such that either x equals three or x is greater than two”, or “some x is either equal to three or greater than two”.

Existential quantifiers can be restricted, by specifying a range for the variable x . For example, the sentence

$$(\exists x \in \mathbb{R})(x = 3 \vee x > 2)$$

says that “there exists a real number x such that x equals three or x is greater than two”, or “some real number is either equal to three or greater than two”.

The sentence to which the existential quantifier applies is the *scope* of the quantifier. *The scope of a quantifier must be enclosed in parenthesis.* If there is no left parenthesis immediately to the right of a quantifier ($\exists x$) or ($\exists x \in U$), then the scope of the quantifier is taken to be the smallest sentence to the right of the quantifier.

For example, let us compare the sentences

$$(3) \quad (\exists x \in \mathbb{R})(x = 3 \vee x > 2).$$

and

$$(4) \quad (\exists x \in \mathbb{R})x = 3 \vee x > 2.$$

In (3), the scope of the quantifier is “ $x = 3 \vee x > 2$ ”, so the sentence is read as “there exists a real number x such that either x equals three or x is greater than two”. In (4), the scope of the quantifier is “ $x = 3$ ”, so the sentence is read as “either there exists a real number which is equal to three, or x is greater than two”. Once again, you should notice that **misplaced parentheses or missing parentheses can make a huge difference.**

2 Homework assignment No. 2, due on Thursday February 4

Book, pages 26-27-28, problems 1 (non-starred items), 3, 5 (non-starred items), 6 (non-starred items), 8 (non-starred items), 10 (non-starred items), 11.